Fractal-Based Point Processes

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Appendix A
Derivations

A.1 POINT PROCESSES: DEFINITION AND MEASURES

A.1.1 Detrended fluctuation analysis for renewal process

To prove Eq. (3.25), we begin by normalizing the sequence of interevent intervals $\{\tau_n\}$. The detrending process removes linear trends from the summed series $\{y_n\}$, which is tantamount to removing the mean from the interevent intervals themselves. With the mean value of the interevent intervals $\mathrm{E}[\tau]$ rendered irrelevant, we set it to zero for algebraic convenience. Furthermore, we divide by the standard deviation to generate a zero-mean unit-variance sequence $\{x_n\}$ by defining

$$x_n \equiv \frac{\tau_n - \mathrm{E}[\tau]}{\sigma_\tau}. \tag{A.1}$$

Without loss of generality we consider $\{x_n\}$ instead of $\{\tau_n\}$ in the following.

For an arbitrary set of ordered pairs $\{(t_n, y_n)\}$, $1 \le n \le k$, classical statistical theory (Press, Teukolsky, Vetterling & Flannery, 1992) yields the residual errors after subtracting the trends. If we define

$$\begin{aligned} z_n &= a t_n + b \\ w_n &= y_n - z_n \end{aligned} \tag{A.2}$$

and

$$\chi^2 = \sum_{n=1}^{k} w_n^2 \tag{A.3}$$

(see Fig. 3.4), and find a and b that minimize χ^2, we obtain

$$\chi^2 = \mathcal{S}_{yy} + \frac{\mathcal{S}_{tt}\mathcal{S}_y^2 + \mathcal{S}\mathcal{S}_{ty}^2 - 2\mathcal{S}_t\mathcal{S}_y\mathcal{S}_{ty}}{\mathcal{S}\mathcal{S}_{tt} - \mathcal{S}_t^2}, \tag{A.4}$$

where we have used the notation of Press et al. (1992) and defined

$$\begin{array}{lll} \mathcal{S} \equiv \sum_{n=1}^{k} 1 & \mathcal{S}_t \equiv \sum_{n=1}^{k} t_n & \mathcal{S}_{tt} \equiv \sum_{n=1}^{k} t_n^2 \\ \mathcal{S}_y \equiv \sum_{n=1}^{k} y_n & \mathcal{S}_{ty} \equiv \sum_{n=1}^{k} t_n y_n & \mathcal{S}_{yy} \equiv \sum_{n=1}^{k} y_n^2. \end{array} \tag{A.5}$$

Substituting $t_n = n$ into Eq. (A.5) we obtain

$$\begin{aligned} \mathcal{S} &= \textstyle\sum_{n=1}^{k} 1 = k \\ \mathcal{S}_t &= \textstyle\sum_{n=1}^{k} t_n = \sum_{n=1}^{k} n = k(k+1)/2 \\ \mathcal{S}_{tt} &= \textstyle\sum_{n=1}^{k} t_n^2 = \sum_{n=1}^{k} n^2 = k(k+1)(2k+1)/6 \\ \mathcal{S}_{ty} &= \textstyle\sum_{n=1}^{k} t_n y_n = \sum_{n=1}^{k} n y_n. \end{aligned} \tag{A.6}$$

To establish the link with detrended fluctuation analysis, we need to substitute n for t_n and take the expectation of Eq. (A.4). (We cannot take the expectation of y_n

and then perform a least-squares fit since $\mathrm{E}[y_n] = 0$ by construction; rather, the fit must precede the expectation.) Taking this expectation yields

$$\mathrm{E}\big[\chi^2\big] = \mathrm{E}[\mathcal{S}_{yy}] + \frac{\mathcal{S}_{tt}\,\mathrm{E}\big[\mathcal{S}_y^2\big] + \mathcal{S}\,\mathrm{E}\big[\mathcal{S}_{ty}^2\big] - 2\mathcal{S}_t\,\mathrm{E}[\mathcal{S}_y\mathcal{S}_{ty}]}{\mathcal{S}\mathcal{S}_{tt} - \mathcal{S}_t^2}, \tag{A.7}$$

where the deterministic nature of t_n permits us to move sums not involving y_n outside the expectations.

We now proceed to simplify the expectations in Eq. (A.7). The key step involves the independence of the sequence $\{x_n\}$. In particular, consider the expectation of the product of two terms y_m and y_n, with $m < n$; we have

$$\begin{aligned}
\mathrm{E}[y_m y_n] &= \mathrm{E}\left[\sum_{p=1}^{m}\sum_{q=1}^{n} x_p x_q\right] \\
&= \sum_{p=1}^{m}\sum_{q=1}^{n} \mathrm{E}[x_p x_q] \\
&= \sum_{p=1}^{m} \{\mathrm{E}[x_p x_q] - \mathrm{E}[x_p]\,\mathrm{E}[x_q]\} + \sum_{p=1}^{m}\sum_{\substack{q=1\\ q\neq p}}^{n} \mathrm{E}[x_p]\;\mathrm{E}[x_q] \\
&= \sum_{p=1}^{m} \{1 - 0\times 0\} + \sum_{p=1}^{m}\sum_{\substack{q=1\\ q\neq p}}^{n} 0\times 0 \\
&= m,
\end{aligned} \tag{A.8}$$

and the reason for constructing a zero-mean unit-variance sequence $\{x_n\}$ now becomes apparent. In general, we have $\mathrm{E}[y_m y_n] = \min(m,n)$, the smaller of m and n. Employing Eq. (A.8) and the last line of Eq. (A.6), we obtain

$$\begin{aligned}
\mathrm{E}[\mathcal{S}_{yy}] &= \mathrm{E}\left[\sum_{n=1}^{k} y_n^2\right] = \sum_{n=1}^{k}\mathrm{E}\big[y_n^2\big] = \sum_{n=1}^{k} n \\
&= k(k+1)/2
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
\mathrm{E}\big[\mathcal{S}_y^2\big] &= \mathrm{E}\left[\sum_{m=1}^{k}\sum_{n=1}^{k} y_m y_n\right] = \sum_{m=1}^{k}\sum_{n=1}^{k}\mathrm{E}[y_m\, y_n] = \sum_{m=1}^{k}\sum_{n=1}^{k}\min(m,n) \\
&= k(k+1)(2k+1)/6
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
\mathrm{E}[\mathcal{S}_y\mathcal{S}_{ty}] &= \sum_{m=1}^{k}\sum_{n=1}^{k} m\,\min(m,n) \\
&= k(k+1)(5k^2+5k+2)/24
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\mathrm{E}\big[\mathcal{S}_{ty}^2\big] &= \sum_{m=1}^{k}\sum_{n=1}^{k} m\,n\,\min(m,n) \\
&= k(k+1)(2k+1)(2k^2+2k+1)/30.
\end{aligned} \tag{A.12}$$

Finally, substituting Eqs. (A.12) and (A.6) into Eq. (A.7), after a fair amount of algebra we obtain

$$\mathrm{E}\big[\chi^2\big] = (k^2 - 4)/15. \tag{A.13}$$

Normalizing Eq. (A.13) by the number of values k, generalizing to arbitrary variance, and taking the square root yields the final result presented in Eq. (3.25).

A.2 POINT PROCESSES: EXAMPLES

A.2.1 Moments for renewal process

In this section we derive expressions for the count probabilities and moments of a stationary renewal point process $dN(t)$.

Let s be any time selected independently of $dN(t)$, and recall that the random variable $\vartheta(s)$ denotes the time remaining between s and the next event in $dN(t)$. We reiterate Eq. (3.12), which provides

$$\begin{aligned} p_\vartheta(s) &= \left[1 - P_\tau(s)\right]/\mathrm{E}[\tau] \\ &= \mathrm{E}[\mu] \int_s^\infty p_\tau(u)\, du. \end{aligned} \tag{A.14}$$

The associated characteristic function becomes

$$\begin{aligned} \phi_\vartheta(\omega) &\equiv \int_0^\infty p_\vartheta(t)\, e^{-i\omega t}\, dt \\ &= \mathrm{E}[\mu] \int_{t=0}^\infty \int_{v=t}^\infty p_\tau(v)\, e^{-i\omega t}\, dv\, dt \\ &= \mathrm{E}[\mu] \int_{v=0}^\infty p_\tau(v) \int_{t=0}^v e^{-i\omega t}\, dt\, dv \\ &= (i\omega)^{-1}\, \mathrm{E}[\mu] \int_{v=0}^\infty p_\tau(v) \left[1 - e^{-i\omega v}\right] dv \\ &= (i\omega)^{-1}\, \mathrm{E}[\mu] \big[1 - \phi_\tau(\omega)\big]. \end{aligned} \tag{A.15}$$

To simplify the notation, we can set $s = 0$ without loss of generality for a stationary renewal point process, thereby permitting the use of $Z(T)$ instead of $N(t)$. With this Ansatz, consider the probability density for the nth event following the origin occurring at a time T. As a result of the renewal nature of $dN(t)$, this becomes $p_\tau(v)$ convolved with itself n times, all convolved with $p_\vartheta(v)$ as the time to the first event. The integral of this probability density yields the probability that the nth event occurs by the time T, which is equivalent to the probability that at least n events have occurred in $Z(T)$.

We thus arrive at

$$\Pr\{Z(T) > n\} = \int_0^T p_\vartheta \star p_\tau^{\star n}(t)\, dt, \tag{A.16}$$

so that

$$\begin{aligned}\Pr\{Z(T)=n\} &= \Pr\{Z(T)>n-1\} - \Pr\{Z(T)>n\} \\ &= \begin{cases} 0 & n<0 \\ 1-\int_0^T p_\vartheta(t)\,dt & n=0 \\ \int_{0-}^T p_\vartheta \star \big[p_\tau^{\star(n-1)} - p_\tau^{\star n}\big](t)\,dt & n>0, \end{cases} \end{aligned} \tag{A.17}$$

which is the probability distribution for the number of counts $Z(T)$. We have used the notation $0-$ for the lower limit in Eq. (A.17) and subsequently to explicitly include delta functions that may occur at $t=0$.

Suppose we now carry out a Fourier transform on Eq. (A.17):

$$\begin{aligned} f_1(\omega,n) &\equiv \int_0^\infty e^{-i\omega T}\,\Pr\{Z(T)=n\}\,dT \\ &= \begin{cases} 0 & n<0 \\ (i\omega)^{-1} - \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big] & n=0 \\ \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big]^2\,\phi_\tau^{n-1}(\omega) & n>0, \end{cases} \end{aligned} \tag{A.18}$$

and following this take the z transform, which yields a time and event-number generating function $f_2(\omega,z)$:

$$\begin{aligned} f_2(\omega,z) &\equiv \sum_{n=0}^\infty f_1(\omega,n)\,z^{-n} \\ &= (i\omega)^{-1} - \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big] \\ &\quad + \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big]^2 \sum_{n=1}^\infty z^{-n}\,\phi_\tau^{n-1}(\omega) \\ &= (i\omega)^{-1} - \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big] \\ &\quad + \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big]^2 z^{-1} \sum_{m=0}^\infty \big[\phi_\tau(\omega)/z\big]^m \\ &= (i\omega)^{-1} - \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big] \\ &\quad + \mathrm{E}[\mu]\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big]^2\,[z-\phi_\tau(\omega)]^{-1}. \end{aligned} \tag{A.19}$$

Next, take k derivatives with respect to z

$$\frac{\partial^k}{\partial z^k}\,f_2(\omega,z) = (-1)^k\,\mathrm{E}[\mu]\,k!\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big]^2\,[z-\phi_\tau(\omega)]^{-(k+1)}. \tag{A.20}$$

Set $z=1$ to obtain

$$(-1)^k\,\frac{\partial^k}{\partial z^k} f_2(\omega,z)\Big|_{z=1} = \mathrm{E}[\mu]\,k!\,(i\omega)^{-2}\big[1-\phi_\tau(\omega)\big]^{1-k}, \tag{A.21}$$

and carry out an inverse Fourier transform. This yields

$$\begin{aligned}
&\int_{\omega=-\infty}^{\infty} e^{+i\omega T}\left[(-1)^k \frac{\partial^k}{\partial z^k} f_2(\omega,z)\Big|_{z=1}\right] \frac{d\omega}{2\pi} \\
&= \int_{\omega=-\infty}^{\infty} e^{+i\omega T}\, \mathrm{E}[\mu]\, k!\, (i\omega)^{-2} \big[1-\phi_\tau(\omega)\big]^{1-k} \frac{d\omega}{2\pi} \\
&= \mathrm{E}[\mu]\, k! \int_{t=0-}^{T} \int_{v=0-}^{t} G^{\star(k-1)}(v)\, \mathrm{E}^{1-k}[\mu]\, dv\, dt \\
&= \mathrm{E}^{2-k}[\mu]\, k! \int_{0-}^{T} (T-t)\, G^{\star(k-1)}(t)\, dt, \qquad \text{(A.22)}
\end{aligned}$$

where we have made use of Eq. (4.15), and again use the notation $0-$ to include delta functions.

However, the two Fourier transforms cancel, so we also obtain

$$\begin{aligned}
&\int_{\omega=-\infty}^{\infty} e^{+i\omega T}\left[(-1)^k \frac{\partial^k}{\partial z^k} f_2(\omega,z)\Big|_{z=1}\right] \frac{d\omega}{2\pi} \\
&= (-1)^k \frac{\partial^k}{\partial z^k} \sum_{n=0}^{\infty} \Pr\{Z(T)=n\}\, z^{-n}\Big|_{z=1} \\
&= \sum_{n=1}^{\infty} \Pr\{Z(T)=n\}\, \frac{(n+k-1)!}{(n-1)!}\, z^{-(n+k)}\Big|_{z=1} \\
&= \mathrm{E}\left\{\frac{[Z(T)+k-1]!}{[Z(T)-1]!}\right\}. \qquad \text{(A.23)}
\end{aligned}$$

Equating Eqs. (A.22) and (A.23) yields the result provided in Eq. (4.19):

$$\mathrm{E}\left\{\frac{[Z(T)+k-1]!}{[Z(T)-1]!}\right\} = \mathrm{E}^{2-k}[\mu]\, k! \int_{0-}^{T} (T-t)\, G^{\star(k-1)}(t)\, dt. \qquad \text{(A.24)}$$

A.3 PROCESSES BASED ON FRACTIONAL BROWNIAN MOTION

A.3.1 Fractal lognormal noise

Following the method employed by Lowen et al. (1997b), we derive expressions for the moments and autocorrelation of fractal lognormal noise. To simplify notation, we reiterate the first and second cumulants:

$$\begin{aligned}
C_1 &\equiv \mathrm{E}[X] \\
C_2 &\equiv \mathrm{Var}[X].
\end{aligned} \qquad \text{(A.25)}$$

For the moments of μ we have

$$\mathrm{E}[\mu^n] = \int_0^\infty p_\mu(y)\, y^n\, dy$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} p_X(x)\, \exp(nX)\, dx \\
&= \int_{-\infty}^{\infty} (2\pi C_2)^{-1/2} \exp\left[nx - \frac{(x-C_1)^2}{2C_2}\right] dx \\
&= (2\pi C_2)^{-1/2} \int_{-\infty}^{\infty} \exp\left[nC_1 + n^2\frac{C_2}{2} - \frac{[x-(C_1+nC_2)]^2}{2C_2}\right] dx \\
&= \exp(nC_1 + n^2 C_2/2)\,(2\pi C_2)^{-1/2} \int_{-\infty}^{\infty} \exp(-z^2/2C_2)\, dz \\
&= \exp(nC_1 + n^2 C_2/2),
\end{aligned} \tag{A.26}
$$

in agreement with Eq. (6.21).

For the autocorrelation of the rate, we write

$$
\begin{aligned}
R_\mu(t) &\equiv \mathrm{E}[\mu(s)\,\mu(s+t)] \\
&= \mathrm{E}\big\{\exp[X(s)]\,\exp[X(s+t)]\big\} \\
&= \mathrm{E}\big\{\exp[X(s)+X(s+t)]\big\}.
\end{aligned} \tag{A.27}
$$

To proceed, we divide $X(s+t)$ into two parts, one proportional to $X(s)$ and one uncorrelated with it

$$
X(s+t) - C_1 = f(t)\,[X(s) - C_1] + X_\perp(s,t). \tag{A.28}
$$

Since $X(s)$ is a Gaussian process, so too is $X_\perp(s,t)$, and since the two are uncorrelated, they are also independent. Substituting Eq. (A.28) into Eq. (A.27) then yields

$$
\begin{aligned}
R_\mu(t) &= \mathrm{E}\big\{\exp[X(s)+X(s+t)]\big\} \\
&= \mathrm{E}\Big\{\exp\big[X(s) + C_1 + f(t)\,[X(s)-C_1] + X_\perp(s,t)\big]\Big\} \\
&= \mathrm{E}\Big[\exp\big\{[1+f(t)]\,X(s)\big\}\Big]\,\mathrm{E}\Big[\exp\big\{[1-f(t)]\,C_1\big\}\Big] \\
&\quad \times \mathrm{E}\Big[\exp\big\{X_\perp(s,t)\big\}\Big],
\end{aligned} \tag{A.29}
$$

where the last step leading to Eq. (A.29) derives from the independence of $X(s)$ and $X_\perp(s,t)$.

Next we find expressions for $f(t)$ as well as for the mean and variance of $X_\perp(s,t)$. Taking the expectation of Eq. (A.28) yields

$$
\begin{aligned}
C_1 - C_1 &= f(t)\,[C_1 - C_1] + \mathrm{E}[X_\perp(s,t)] \\
\mathrm{E}[X_\perp(s,t)] &= 0.
\end{aligned} \tag{A.30}
$$

We then multiply Eq. (A.28) by $X(s) - C_1$ and take expectations,

$$
\begin{aligned}
&\mathrm{E}\big\{[X(s+t)-C_1]\,[X(s)-C_1]\big\} \\
&\qquad = f(t)\,\mathrm{E}\big\{[X(s)-C_1]\,[X(s)-C_1]\big\} + \mathrm{E}\big\{X_\perp(s,t)\,[X(s)-C_1]\big\} \\
R_X(t) - C_1^2 &= f(t)\,C_2 + C_1\,\mathrm{E}[X_\perp(s,t)] \\
f(t) &= \left[R_X(t) - C_1^2\right]/C_2\,,
\end{aligned} \tag{A.31}
$$

where we have made use of the independence of $X(s)$ and $X_\perp(s,t)$, as well as Eq. (A.30). Rearranging Eq. (A.28) yields the variance of $X_\perp(s,t)$:

$$\begin{aligned} X_\perp(s,t) &= [X(s+t) - C_1] - f(t)\,[X(s) - C_1] \\ X_\perp^2(s,t) &= [X(s+t) - C_1]^2 + f^2(t)\,[X(s) - C_1]^2 \\ &\quad - 2f(t)\,[X(s) - C_1]\,[X(s+t) - C_1] \\ \mathrm{E}\big[X_\perp^2(s,t)\big] &= C_2 + f^2(t)\,C_2 - 2f(t)\,\big[R_X(t) - C_1^2\big] \\ \mathrm{Var}[X_\perp(s,t)] &= \big[1 - f^2(t)\big]C_2, \end{aligned} \tag{A.32}$$

where we have made use of Eq. (A.31).

Finally, we substitute Eqs. (A.30)–(A.32) into Eq. (A.29). To evaluate the ensuing expressions, consider Eq. (A.26). Any fixed expression can substitute for n in the second line, so that

$$\begin{aligned} R_\mu(t) &= \mathrm{E}\Big[\exp\big\{[1+f(t)]\,X(s)\big\}\Big]\,\mathrm{E}\Big[\exp\big\{[1-f(t)]\,C_1\big\}\Big] \\ &\quad \times \mathrm{E}\Big[\exp\big\{X_\perp(s,t)\big\}\Big] \\ &= \exp\big\{[1+f(t)]\,C_1 + [1+f(t)]^2\,C_2/2\big\}\,\exp\big\{[1-f(t)]\,C_1\big\} \\ &\quad \times \exp\big\{\big[1-f^2(t)\big]C_2/2\big\} \\ R_\mu(t) &= \exp\big\{2C_1 + [1+f(t)]\,C_2\big\} \\ &= \exp\big\{2[C_1 + C_2/2]\big\}\exp\big\{f(t)\,C_2\big\} \\ &= \mathrm{E}^2[\mu]\,\exp\big\{R_X(t) - \mathrm{E}^2[X]\big\}, \end{aligned} \tag{A.33}$$

in agreement with Eq. (6.22).

A.4 FRACTAL RENEWAL PROCESSES

In preparation for the results that follow, we first obtain expressions for three quantities involving the square root of negative unity. Using the De Moivre relation for $\theta = \pi/2$ provides

$$\begin{aligned} \exp(i\pi/2) &= \cos(\pi/2) + i\sin(\pi/2) \\ &= i. \end{aligned} \tag{A.34}$$

Raising both sides of Eq. (A.34) to the same power yields

$$\begin{aligned} i^x &= \exp(ix\pi/2) \\ \mathrm{Re}\{i^x\} &= \cos(x\pi/2). \end{aligned} \tag{A.35}$$

The second expression is obtained by taking logarithm of both sides of Eq. (A.34):

$$\ln(i) = i\pi/2. \tag{A.36}$$

We can generally ignore multiplicative factors inside the logarithm when the argument otherwise assumes a large or small value. This is demonstrated via

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(c\,x)}{\ln(x)} &= \lim_{x\to 0}\frac{\ln(c)+\ln(x)}{\ln(x)} \\ &= 1+\lim_{x\to 0}\frac{\ln(c)}{\ln(x)} \\ &= 1, \end{aligned} \qquad \text{(A.37)}$$

so that $\ln(cx) \approx \ln(x)$ as $x \to 0$ for any finite value c. A similar result obtains in the limit $x \to \infty$. Exceptions occur in cases where we must distinguish two forms that would otherwise appear identical, such as those in Eqs. (7.18) and (7.19) for $\gamma = 2$.

Third, we derive an expression for the square root of a complex number:

$$\begin{aligned}\sqrt{a+ib} &= c+id \\ a+ib &= c^2-d^2+i2cd \\ a &= c^2-d^2 \\ b &= 2cd \\ a^2+b^2 &= c^4+d^4-2c^2d^2+4c^2d^2=(c^2+d^2)^2 \\ \sqrt{a^2+b^2} &= c^2+d^2 \\ \sqrt{a^2+b^2}+a &= 2c^2 \\ \sqrt{a^2+b^2}-a &= 2d^2\end{aligned}$$

$$\begin{aligned} c &= \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} \\ d &= \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}. \end{aligned} \qquad \text{(A.38)}$$

Thus, the quantity $c + id$ forms a solution to $\sqrt{a+ib}$.

A.4.1 Spectrum in the mid-frequency range

In this section we obtain approximate expressions for the spectrum of the fractal renewal point process, as well as for the symmetric alternating fractal renewal process, in the mid-frequency range, $B^{-1} \ll f \ll A^{-1}$.

Since the power-law tail of the interevent-interval probability density function determines the fractal behavior of the process, the results are insensitive to the precise form of the density. We make use of the abrupt-cutoff power-law form to simplify the calculations; similar results would result when using any power-law-varying density. To simplify the notation, we express the results in terms of the radian frequency $\omega \equiv 2\pi f$; this does not, of course, alter the validity of the arguments.

We begin with Eq. (7.3), the characteristic function for the abrupt-cutoff power-law probability density provided in Eq. (7.1). Substituting $y \equiv x/i\omega A$, we have

$$\begin{aligned} \phi_\tau(\omega) &= \frac{\gamma}{1-(A/B)^\gamma} \int_1^{B/A} e^{-i\omega A y}\, y^{-(\gamma+1)}\, dy \\ &\to \frac{\gamma}{1-(A/B)^\gamma} \int_1^{B/A} e^{0}\, y^{-(\gamma+1)}\, dy \qquad \text{(A.39)} \\ &= \frac{\gamma}{1-(A/B)^\gamma} \int_1^{B/A} y^{-(\gamma+1)}\, dy \\ &= \frac{\gamma}{1-(A/B)^\gamma} \frac{1-(B/A)^{-\gamma}}{\gamma} \\ &= 1, \qquad \text{(A.40)} \end{aligned}$$

where Eq. (A.39) derives from the mid-frequency assumption $\omega/2\pi = f \ll A^{-1}$. Thus, to zeroth order we have $\phi_\tau(\omega) \approx 1$.

Use of this approximation leads to

$$\mathrm{Re}\left\{\frac{1+\phi_\tau(\omega)}{1-\phi_\tau(\omega)}\right\} \approx \mathrm{Re}\left\{\frac{1+1}{1-\phi_\tau(\omega)}\right\} = 2\,\frac{\mathrm{Re}\{1-\phi_\tau(\omega)\}}{\left|1-\phi_\tau(\omega)\right|^2} \qquad \text{(A.41)}$$

for the nontrivial part of the point-process spectrum, and to

$$\mathrm{Re}\left\{\frac{1-\phi_\tau(\omega)}{1+\phi_\tau(\omega)}\right\} \approx \mathrm{Re}\left\{\frac{1-\phi_\tau(\omega)}{1+1}\right\} = \tfrac{1}{2}\mathrm{Re}\{1-\phi_\tau(\omega)\} \qquad \text{(A.42)}$$

for the symmetric alternating fractal renewal process. To obtain asymptotic expressions, we expand $1-\phi_\tau(\omega)$ into a series of powers of ω until we obtain the first term with a nonzero real part. Substituting these results into Eqs. (A.41) and (A.42) yields Eqs. (7.8) and (8.11), respectively, as we will show forthwith.

To carry the calculation forward, we employ Eq. (7.3) evaluated at $\omega = 0$ and subtract Eq. (7.3) again:

$$1-\phi_\tau(\omega) = \gamma(i\omega\, A)^\gamma \left[1-(A/B)^\gamma\right]^{-1} \int_{i\omega A}^{i\omega B} (1-e^{-x})\, x^{-(\gamma+1)}\, dx. \qquad \text{(A.43)}$$

Since $B^{-1} \ll f = \omega/2\pi \ll A^{-1}$, we have $A/B \to 0$ and $\omega B \to \infty$. Defining $z \equiv i\omega A$, we therefore obtain

$$\begin{aligned} 1-\phi_\tau(\omega) &\to \gamma z^\gamma \int_z^\infty (1-e^{-x})\, x^{-(\gamma+1)}\, dx \\ &= \gamma \int_1^\infty (1-e^{-zy})\, y^{-(\gamma+1)}\, dy. \qquad \text{(A.44)} \end{aligned}$$

We proceed to calculate express results for various ranges of the exponent γ.

- For $0 < \gamma < 1$ we make use of l'Hôpital's rule to evaluate the limit

$$\begin{aligned}\lim_{z\to 0}\frac{1-\phi_\tau(\omega)}{z^\gamma} &= \lim_{z\to 0}\frac{\gamma\displaystyle\int_1^\infty (1-e^{-zy})\,y^{-(\gamma+1)}\,dy}{z^\gamma}\\ &= \lim_{z\to 0}\frac{\gamma\displaystyle\int_1^\infty e^{-zy}\,y^{-\gamma}\,dy}{\gamma z^{\gamma-1}}\\ &= \lim_{z\to 0}\frac{\gamma z^{\gamma-1}\displaystyle\int_z^\infty e^{-x}\,x^{-\gamma}\,dx}{\gamma z^{\gamma-1}}\\ &= \int_0^\infty e^{-x}\,x^{-\gamma}\,dx\\ &= \Gamma(1-\gamma),\end{aligned} \tag{A.45}$$

so that

$$\begin{aligned}1-\phi_\tau(\omega) &\to \Gamma(1-\gamma)\,(i\omega A)^\gamma\\ |1-\phi_\tau(\omega)| &\to \Gamma(1-\gamma)\,(\omega A)^\gamma\\ \mathrm{Re}\big\{1-\phi_\tau(\omega)\big\} &\to \Gamma(1-\gamma)\,\cos(\pi\gamma/2)\,(\omega A)^\gamma.\end{aligned} \tag{A.46}$$

- For $\gamma = 1$, we again use l'Hôpital's rule and evaluate

$$\begin{aligned}\lim_{z\to 0}\frac{1-\phi_\tau(\omega)}{-z\ln(z)} &= \lim_{z\to 0}\frac{\displaystyle\int_z^\infty (1-e^{-x})\,x^{-2}\,dx}{-\ln(z)}\\ &= \lim_{z\to 0}\frac{-(1-e^{-z})\,z^{-2}}{-1/z}\\ &= \lim_{z\to 0}\frac{1-e^{-z}}{z}\\ &= 1,\end{aligned} \tag{A.47}$$

which, with the help of the results set forth at the beginning of Sec. A.4, leads to

$$\begin{aligned}1-\phi_\tau(\omega) &\to -(i\omega A)\,\ln(i\omega A)\\ &= -(i\omega A)\,\ln(i) - (i\omega A)\,\ln(\omega A)\\ &= (\pi/2)(\omega A) - i(\omega A)\,\ln(\omega A)\\ |1-\phi_\tau(\omega)| &\to -(\omega A)\,\ln(\omega A)\\ \mathrm{Re}\big\{1-\phi_\tau(\omega)\big\} &\to (\pi/2)(\omega A).\end{aligned} \tag{A.48}$$

- For $\gamma > 1$ in general, the dominant term becomes linear in ω. The limit

$$\begin{aligned}
\lim_{z\to 0} \frac{1-\phi_\tau(\omega)}{z} &= \lim_{z\to 0} \frac{\gamma \int_1^\infty (1-e^{-zy})\, y^{-(\gamma+1)}\, dy}{z} \\
&= \lim_{z\to 0} \frac{\gamma \int_1^\infty e^{-zy}\, y^{-\gamma}\, dy}{1} \\
&= \gamma \int_1^\infty y^{-\gamma}\, dy \\
&= \frac{\gamma}{\gamma-1} \qquad \text{(A.49)}
\end{aligned}$$

implies that

$$\begin{aligned}
1-\phi_\tau(\omega) &\to \frac{\gamma}{\gamma-1}\, i\omega A \\
|1-\phi_\tau(\omega)| &\to \frac{\gamma}{\gamma-1}\, \omega A, \qquad \text{(A.50)}
\end{aligned}$$

for all $\gamma > 1$. To evaluate the spectrum we still require a first term with a nonzero real part. We continue by expanding the quantity

$$1-\phi_\tau(\omega) - \frac{\gamma}{\gamma-1}\, i\omega A. \qquad \text{(A.51)}$$

For $1 < \gamma < 2$, the next term arises from the limit

$$\begin{aligned}
&\lim_{z\to 0} \frac{1-\phi_\tau(\omega) - \gamma z/(\gamma-1)}{z^\gamma} \\
&= \lim_{z\to 0} \frac{\gamma \int_1^\infty (1-e^{-zy})\, y^{-(\gamma+1)}\, dy - \gamma z/(\gamma-1)}{z^\gamma} \\
&= \lim_{z\to 0} \frac{\gamma \int_1^\infty e^{-zy}\, y^{-\gamma}\, dy - \gamma/(\gamma-1)}{\gamma z^{\gamma-1}} \\
&= \lim_{z\to 0} \frac{-\gamma \int_1^\infty e^{-zy}\, y^{1-\gamma}\, dy}{\gamma(\gamma-1)\, z^{\gamma-2}} \\
&= \lim_{z\to 0} \frac{-\gamma z^{\gamma-2} \int_z^\infty e^{-x}\, x^{1-\gamma}\, dx}{\gamma(\gamma-1)\, z^{\gamma-2}} \\
&= -(\gamma-1)^{-1} \int_0^\infty e^{-x}\, x^{1-\gamma}\, dx \\
&= -(\gamma-1)^{-1}\, \Gamma(2-\gamma), \qquad \text{(A.52)}
\end{aligned}$$

so that

$$\mathrm{Re}\big\{1-\phi_\tau(\omega)\big\} \to (\gamma-1)^{-1}\,\Gamma(2-\gamma)\,\big[-\cos(\pi\gamma/2)\big]\,(\omega A)^\gamma. \qquad \text{(A.53)}$$

- For $\gamma = 2$ we again obtain logarithmic correction terms,

$$\begin{aligned}
\lim_{z\to 0}\frac{1-\phi_\tau(\omega)-\gamma z/(\gamma-1)}{z^\gamma\,\ln(z)} &= \lim_{z\to 0}\frac{2z^2\displaystyle\int_z^\infty (1-e^{-x})\,x^{-3}\,dx - 2z}{z^2\,\ln(z)} \\
&= \lim_{z\to 0}\frac{\displaystyle\int_z^\infty (1-e^{-x})\,x^{-3}\,dx - z^{-1}}{\ln(z)/2} \\
&= \lim_{z\to 0}\frac{-(1-e^{-z})\,z^{-3}+z^{-2}}{z^{-1}/2} \\
&= \lim_{z\to 0}\frac{(z-1+e^{-z})}{z^2/2} \\
&= 1, \qquad \text{(A.54)}
\end{aligned}$$

which results in

$$\mathrm{Re}\big\{1-\phi_\tau(\omega)\big\} \to -(\omega A)^2\ln(\omega A). \qquad \text{(A.55)}$$

- Finally, for $\gamma > 2$ the power-law exponents do not depend on γ, but are constant at the square of ω:

$$\begin{aligned}
&\lim_{z\to 0}\frac{1-\phi_\tau(\omega)-\gamma z/(\gamma-1)}{z^2} \\
&\quad= \lim_{z\to 0}\frac{\gamma\displaystyle\int_1^\infty (1-e^{-zy})\,y^{-(\gamma+1)}\,dy - \gamma z/(\gamma-1)}{z^2} \\
&\quad= \lim_{z\to 0}\frac{\gamma\displaystyle\int_1^\infty e^{-zy}\,y^{-\gamma}\,dy - \gamma/(\gamma-1)}{2z} \\
&\quad= \lim_{z\to 0}\frac{\gamma\displaystyle\int_1^\infty e^{-zy}\,y^{1-\gamma}\,dy}{2} \\
&\quad= -\tfrac{1}{2}\,\gamma\int_1^\infty y^{1-\gamma}\,dy \\
&\quad= -\tfrac{1}{2}\,\gamma\,(2-\gamma)^{-1}, \qquad \text{(A.56)}
\end{aligned}$$

which gives

$$\mathrm{Re}\big\{1-\phi_\tau(\omega)\big\} \to \tfrac{1}{2}\,\gamma(2-\gamma)^{-1}(\omega A)^2. \qquad \text{(A.57)}$$

A.4.2 Spectrum for $\gamma = \frac{1}{2}$

For the particular case $\gamma = \frac{1}{2}$, the smooth-transition interevent-interval probability density function provided in Eq. (7.5) simplifies to

$$p_\tau(t) = \sqrt{A/\pi}\, \exp\Big(2\sqrt{A/B}\Big)\, \exp(-A/t)\, \exp(-t/B)\, t^{-3/2}, \tag{A.58}$$

and the corresponding characteristic function becomes

$$\phi_\tau(\omega) = \exp\big[2\sqrt{A/B} - 2(A/B + i\omega A)^{1/2}\big]. \tag{A.59}$$

The derivative of Eq. (A.59) at $\omega = 0$ yields the mean interevent interval,

$$\begin{aligned} \frac{d}{d\omega}\phi_\tau(\omega) &= \phi_\tau(\omega)\Big[-(A/B + i\omega A)^{-1/2}\Big]\, iA \\ \mathrm{E}[\tau] = i\,\frac{d}{d\omega}\phi_\tau(\omega)\Big|_{\omega=0} &= \sqrt{AB}. \end{aligned} \tag{A.60}$$

For simplicity, we set the mean interval to unity, so that $AB = 1$. Equation (A.59) then becomes

$$\begin{aligned} \phi_\tau(\omega) &= \exp\Big[2A - 2\left(A^2 + i\omega A\right)^{1/2}\Big] \\ &= \exp\Big[2A - \sqrt{2A}\left(\sqrt{A^2+\omega^2} + A\right)^{1/2} \\ &\qquad - i\sqrt{2A}\left(\sqrt{A^2+\omega^2} - A\right)^{1/2}\Big] \\ &= \exp(-c)\,\big[\cos(d) - i\sin(d)\big], \end{aligned} \tag{A.61}$$

where we have defined

$$c \equiv \sqrt{2A}\left(\sqrt{A^2+\omega^2} + A\right)^{1/2} - 2A \tag{A.62}$$

$$d \equiv \sqrt{2A}\left(\sqrt{A^2+\omega^2} - A\right)^{1/2}, \tag{A.63}$$

making use of Eq. (A.38) set out at the beginning of Sec. A.4.

Reiterating Eq. (4.16) and substituting Eq. (A.61) into it yields

$$\begin{aligned} &S_N(f) - \mathrm{E}^2[\mu]\,\delta(f) \\ &\quad= \mathrm{E}[\mu]\,\mathrm{Re}\left[\frac{1+\phi_\tau(2\pi f)}{1-\phi_\tau(2\pi f)}\right] \\ &\quad= \mathrm{Re}\left\{\frac{1+e^{-c}\big[\cos(d) - i\sin(d)\big]}{1-e^{-c}\big[\cos(d) - i\sin(d)\big]} \times \frac{e^{c} - \big[\cos(d) + i\sin(d)\big]}{e^{c} - \big[\cos(d) + i\sin(d)\big]}\right\} \\ &\quad= \mathrm{Re}\left\{\frac{e^{c} - 2i\sin(d) - e^{-c}\big[\cos^2(d) + \sin^2(d)\big]}{e^{c} - 2\cos(d) + e^{-c}\big[\cos^2(d) + \sin^2(d)\big]}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{c}-e^{-c}}{e^{c}-2\cos(d)+e^{-c}} \\
&= \frac{\sinh(c)}{\cosh(c)-\cos(d)},
\end{aligned}
\tag{A.64}
$$

which reproduces Eqs. (7.10)–(7.12).

A.4.3 Coincidence rate in the medium-time limit

Equation (7.13) derives from Eq. (7.8) via the Fourier-transform-pair relations that comprise Eqs. (3.57) and (3.58). Because our focus is on the mid-scale limit, we typically ignore delta functions at zero time or frequency, as well as limits for large times and frequencies.

Results for the regions $0 < \gamma < 1$ and $1 < \gamma < 2$ follow directly from Eqs. (5.44) and (5.45). It thus remains to consider the ranges $\gamma > 2$, $\gamma = 2$, and $\gamma = 1$.

- For $\gamma > 2$, the spectrum remains relatively constant in the limit $1/B \ll f \ll 1/A$, differing little from its low-frequency limit; it therefore resembles the spectrum for a homogeneous Poisson process provided in Eq. (4.9c). The coincidence rate thus follows the form in Eq. (4.9d), which appears in Eq. (7.13) for $\gamma > 2$.

- For $\gamma = 2$, we begin with the coincidence rate given in Eq. (7.13) to obtain

$$
\begin{aligned}
S_N(f) &= 2\mathrm{E}[\mu] \int_A^\infty \cos(2\pi f t)\, \tfrac{1}{4} t^{-1} dt \\
&= \frac{\mathrm{E}[\mu]}{2} \int_{2\pi f A}^\infty \cos(x)\, x^{-1} dx.
\end{aligned}
\tag{A.65}
$$

  The cosine factor in the integrand ensures that the integral converges for large x; in fact, since the integral diverges near $x = 0$ and $2\pi f A \ll 1$, the contribution of this factor for large values of x becomes negligible. For small values of x, the cosine term does not vary significantly, and also becomes unimportant. Bearing these arguments in mind, we then have to first order

$$
\begin{aligned}
S_N(f) &= \frac{\mathrm{E}[\mu]}{2} \int_{2\pi f A}^\infty \cos(x)\, x^{-1} dx \\
&\approx \frac{\mathrm{E}[\mu]}{2} \int_{2\pi f A}^1 x^{-1} dx \\
&= \frac{\mathrm{E}[\mu]}{2} \big[-\ln(2\pi f A)\big].
\end{aligned}
\tag{A.66}
$$

  This agrees with Eq. (7.8), and thus establishes the validity of Eq. (7.13) for $\gamma = 2$.

- For $\gamma = 1$, we employ a similar argument but center it on the forward Fourier transform to obtain

$$\begin{aligned} G(t) &= 2\pi \mathrm{E}[\mu] \int_0^\infty \cos(2\pi f t)\,\big[\ln(2\pi f A)\big]^{-2}(2\pi f A)^{-1}\,df \\ &= \frac{\mathrm{E}[\mu]}{A}\int_0^\infty \cos(x)\,\big[\ln(Ax/t)\big]^{-2} x^{-1} dx \\ &\approx \frac{\mathrm{E}[\mu]}{A}\int_0^1 \cos(x)\,\big[\ln(Ax/t)\big]^{-2} x^{-1} dx \\ &\approx \frac{\mathrm{E}[\mu]}{A}\int_0^1 \big[\ln(Ax/t)\big]^{-2} x^{-1} dx \\ &= \frac{\mathrm{E}[\mu]}{A}\int_{-\infty}^{\ln(A/t)} y^{-2} dy \qquad \text{(A.67)} \\ &= \frac{\mathrm{E}[\mu]}{A}\left[\frac{-1}{\ln(A/t)}\right] \\ &= \mathrm{E}[\mu]\,A^{-1}\big[\ln(t/A)\big]^{-1}, \qquad \text{(A.68)} \end{aligned}$$

which accords with Eq. (7.13) for $\gamma = 1$. Equation (A.67) makes use of the substitution $y = \ln(Ax/t)$.

A.4.4 Normalized variances in the medium-time limit

Proceeding to the normalized variance $F(T)$ in Eq. (7.18), Eqs. (5.44) and (5.45) again provide results for $0 < \gamma < 1$ and $1 < \gamma < 2$. It therefore remains to consider the ranges $\gamma > 2$, $\gamma = 2$, and $\gamma = 1$:

- For $\gamma > 2$, results for the homogeneous Poisson process continue to apply, and we obtain the result provided in Eq. (4.9a).

- For $\gamma = 2$, we employ Eq. (3.55) to obtain

$$G(t) - \mathrm{E}^2[\mu] = \frac{\mathrm{E}[\mu]}{2}\,\frac{d^2}{dT^2}\big[T\ln(T/A)/2\big]_{T=t} = \tfrac{1}{4}\mathrm{E}[\mu]t^{-1}, \qquad \text{(A.69)}$$

in accordance with Eq. (7.13).

- For $\gamma = 1$, a similar approach leads to

$$\begin{aligned} G(t) - \mathrm{E}^2[\mu] &= \frac{\mathrm{E}[\mu]}{2}\,\frac{d^2}{dT^2}\Big\{TA^{-1}\big[\ln(T/A)\big]^{-1}T\Big\}_{T=t} \\ &= \frac{\mathrm{E}[\mu]}{2A}\left[\frac{2}{\ln(t/A)} - \frac{3}{\ln^2(t/A)} + \frac{2}{\ln^3(t/A)}\right] \\ &\approx \frac{\mathrm{E}[\mu]}{2A}\,\frac{2}{\ln(t/A)} \\ &= \mathrm{E}[\mu]\,A^{-1}\big[\ln(t/A)\big]^{-1}, \qquad \text{(A.70)} \end{aligned}$$

which is identical to the result given in Eq. (7.13) for $\gamma = 1$.

Results for the normalized Haar-wavelet variance $A(T)$ in Eq. (7.19) follow directly from Eqs. (5.44), (5.45), and (4.9a), except for $\gamma = 1$ and $\gamma = 2$. We employ Eq. (3.41) for these two cases:

- For $\gamma = 2$, we have

$$\begin{aligned} A(T) &= 2F(T) - F(T) \\ &= \ln(T/A) - \tfrac{1}{2}\ln(2T/A) \\ &= \ln(T/A) - \tfrac{1}{2}\ln(T/A) - \tfrac{1}{2}\ln(2) \\ &= \tfrac{1}{2}\ln(T/A) - \tfrac{1}{2}\ln(2) \\ &= \tfrac{1}{2}\ln(T/2A), \end{aligned} \tag{A.71}$$

  in agreement with Eq. (7.19).

- For $\gamma = 1$, we obtain

$$\begin{aligned} A(T) &= 2F(T) - F(T) \\ &= 2A^{-1}\big[\ln(T/A)\big]^{-1} T - A^{-1}\big[\ln(2T/A)\big]^{-1}(2T) \\ A(T) &= 2A^{-1} T \left[\frac{1}{\ln(T/A)} - \frac{1}{\ln(2T/A)}\right] \\ &= 2A^{-1} T \, \frac{\ln(2T/A) - \ln(T/A)}{\ln(T/A)\, \ln(2T/A)} \\ &= 2A^{-1} T \, \frac{\ln(2)}{\ln(T/A)\, \ln(2T/A)} \\ &\approx 2\ln(2)\, A^{-1}\big[\ln(T/A)\big]^{-2} T, \end{aligned} \tag{A.72}$$

  which also accords with the result provided in Eq. (7.19).

A.5 ALTERNATING FRACTAL RENEWAL PROCESS

A.5.1 Alternating-renewal-process spectrum

We obtain the spectrum of the alternating renewal process by calculating a sequence of quantities, each from the preceding: the probability distribution function of the counting process $N(t)$, the characteristic function of the forward recurrence time, the autocorrelation, and finally the spectrum.

Assume that at $t = 0$ the alternating fractal renewal process lies in state 1, so that $X(0) = 1$. As a first step, we seek the probability that the count exceeds a certain even, nonnegative value $2n$. For this to occur, the interevent interval that encompasses $t = 0$ must end, as well as the next n intervals of both types. The probability that

$N(t) > 2n$ is then the probability that the aggregate of the $2n+1$ intervals does not exceed t. Thus,

$$\begin{aligned} &\Pr\{N(t) > 2n \,|\, X(0) = 1\} \\ &\quad = \Pr\left\{\vartheta_a(0) + \sum_{k=0}^{n-1} [\tau_{ak} + \tau_{bk}] < t \,\middle|\, X(0) = 1\right\} \\ &\quad = \int_0^t p_{\vartheta a}(s) \star p_{\tau a}^{\star n}(s) \star p_{\tau b}^{\star n}(s)\, ds, \end{aligned} \tag{A.73}$$

where ϑ again denotes the forward recurrence time.

We next require an expression for the characteristic function of the forward recurrence time. Employing Eq. (3.12), we have

$$\begin{aligned} \phi_\vartheta(\omega) &= \int_{t=0}^{\infty} \exp(-i\omega t)\, p_\vartheta(t)\, dt \\ &= \int_{t=0}^{\infty} \exp(-i\omega t)\, [1 - P_\tau(t)]\, \mathrm{E}[\mu]\, dt \\ &= \mathrm{E}[\mu] \int_{t=0}^{\infty} \exp(-i\omega t) \int_{u=t}^{\infty} p_\tau(u)\, du\, dt \\ &= \mathrm{E}[\mu] \int_{u=0}^{\infty} p_\tau(u) \int_{t=0}^{u} \exp(-i\omega t)\, dt\, du \\ &= (i\omega)^{-1}\, \mathrm{E}[\mu] \int_{u=0}^{\infty} p_\tau(u) \big[1 - \exp(-i\omega t)\big]\, du \\ &= (i\omega)^{-1}\, \mathrm{E}[\mu] \big[1 - \phi_\tau(\omega)\big]. \end{aligned} \tag{A.74}$$

Taking a Fourier transform of the convolution in Eq. (A.73), and substituting $\omega = 2\pi f$, yields a simple expression involving characteristic functions

$$\begin{aligned} &\int_0^\infty \exp(-i2\pi f t)\, \Pr\{N(t) > 2n \,|\, X(0) = 1\}\, dt \\ &\quad = \int_0^\infty \exp(-i2\pi f t) \int_0^t p_{\vartheta a}(s) \star p_{\tau a}^{\star n}(s) \star p_{\tau b}^{\star n}(s)\, ds\, dt \\ &\quad = \mathrm{E}[\mu_a]\, (i2\pi f)^{-2} \big[1 - \phi_{\tau a}(2\pi f)\big]\, \phi_{\tau a}^n(2\pi f)\, \phi_{\tau b}^n(2\pi f), \end{aligned} \tag{A.75}$$

where one factor of $(i2\pi f)^{-1}$ arises from Eq. (A.74), and the other from the integration in Eq. (A.73). Similarly, for an odd positive number of intervals $2n+1$, we obtain

$$\begin{aligned} &\int_0^\infty \exp(-i2\pi f t)\, \Pr\{N(t) > 2n+1 \,|\, X(0) = 1\}\, dt \\ &\quad = \mathrm{E}[\mu_a]\, (i2\pi f)^{-2} \big[1 - \phi_{\tau a}(2\pi f)\big]\, \phi_{\tau a}^n(2\pi f)\, \phi_{\tau b}^{n+1}(2\pi f). \end{aligned} \tag{A.76}$$

Proceeding to the autocorrelation leads to

$$R_X(t) \equiv \mathrm{E}[X(0)\, X(t)]$$

$$
\begin{aligned}
&= \Pr\{X(0) = 1 \text{ and } X(t) = 1\} \\
&= \Pr\{X(0) = 1\} \Pr\{X(t) = 1 \mid X(0) = 1\} \\
&= \mathrm{E}[X] \Pr\{N(t) \text{ is even}\} \\
&= \mathrm{E}[X] \Pr\left\{\sum_{n=0}^{\infty} N(t) = 2n\right\} \\
&= \mathrm{E}[X] \sum_{n=0}^{\infty} \Pr\{N(t) = 2n\} \\
&= \mathrm{E}[X] \sum_{n=0}^{\infty} \Big(\Pr\{N(t) > 2n-1\} - \Pr\{N(t) > 2n\}\Big) \\
&= \mathrm{E}[X] \left[1 + \sum_{n=0}^{\infty} \Big(\Pr\{N(t) > 2n+1\} - \Pr\{N(t) > 2n\}\Big)\right]. \quad \text{(A.77)}
\end{aligned}
$$

Finally, taking the Fourier transform yields the spectrum

$$
\begin{aligned}
&S_X(f)/\mathrm{E}[X] \\
&\quad = \frac{1}{\mathrm{E}[X]} \int_{-\infty}^{\infty} \exp(-i2\pi f t)\, R_X(t)\, dt \\
&\quad = \frac{2}{\mathrm{E}[X]} \mathrm{Re}\left\{\int_0^{\infty} \exp(-i2\pi f t)\, R_X(t)\, dt\right\} \\
&\quad = 2\mathrm{Re}\left\{\int_0^{\infty} \exp(-i2\pi f t) \left[1 \right.\right. \\
&\qquad\qquad \left.\left. + \sum_{n=0}^{\infty} \Big(\Pr\{N(t) > 2n+1\} - \Pr\{N(t) > 2n\}\Big)\right] dt\right\} \\
&\quad = \delta(f) + 2\mathrm{Re}\left\{\int_0^{\infty} \exp(-i2\pi f t) \right. \\
&\qquad\qquad \left. \times \sum_{n=0}^{\infty} \Big(\Pr\{N(t) > 2n+1\} - \Pr\{N(t) > 2n\}\Big)\, dt\right\} \\
&\quad = \delta(f) + 2\mathrm{Re}\left\{\sum_{n=0}^{\infty} \mathrm{E}[\mu_a]\, (i2\pi f)^{-2} \big[1 - \phi_{\tau a}(\omega)\big] \right. \\
&\qquad\qquad \left. \times \Big[\phi_{\tau a}^{n}(2\pi f)\, \phi_{\tau b}^{n+1}(2\pi f) - \phi_{\tau a}^{n}(2\pi f)\, \phi_{\tau b}^{n}(2\pi f)\Big]\right\} \\
&\quad = \delta(f) + 2\mathrm{E}[\mu_a]\, (2\pi f)^{-2}\, \mathrm{Re}\left\{\big[1 - \phi_{\tau a}(2\pi f)\big] \right. \\
&\qquad\qquad \left. \times \big[1 - \phi_{\tau b}(2\pi f)\big] \sum_{n=0}^{\infty} \phi_{\tau a}^{n}(2\pi f)\, \phi_{\tau b}^{n}(2\pi f)\right\}
\end{aligned}
$$

$$= \quad \delta(f) + \frac{2\mathrm{E}[\mu_a]}{(2\pi f)^2}\,\mathrm{Re}\left\{\frac{\big[1-\phi_{\tau a}(2\pi f)\big]\big[1-\phi_{\tau b}(2\pi f)\big]}{1-\phi_{\tau a}(2\pi f)\,\phi_{\tau b}(2\pi f)}\right\}, \quad \text{(A.78)}$$

which leads directly to Eq. (8.5).

A.5.2 Low-frequency limit of the spectrum

To determine the spectrum as $f \to 0$, we use Eq. (8.5) [or, equivalently, Eq. (A.78)], retaining terms to second order in frequency at each stage. To simplify the notation, we use radian frequency $\omega \equiv 2\pi f$, and the quantities s, t, u, and v to represent fixed constants. We then have

$$\phi_{\tau a}(\omega) \approx 1 - i\omega\mathrm{E}[\tau_a] - \frac{\omega^2}{2}\,\mathrm{E}[\tau_a^2], \quad \text{(A.79)}$$

along with the analogous expression for $\phi_{\tau b}(\omega)$. Substituting Eq. (A.79) into Eq. (8.5), and ignoring the delta function which does not appear in the limit, yields

$$\begin{aligned}
&\big(\mathrm{E}[\tau_a] + \mathrm{E}[\tau_b]\big)\lim_{\omega\to 0} S_X(\omega)/2 \\
&\quad = \lim_{\omega\to 0}\omega^{-2}\,\mathrm{Re}\left\{\frac{\big[1-\phi_{\tau a}(\omega)\big]\,\big[1-\phi_{\tau b}(\omega)\big]}{1-\phi_{\tau a}(\omega)\,\phi_{\tau b}(\omega)}\right\} \\
&\quad = \lim_{\omega\to 0}\mathrm{Re}\left\{\frac{\omega^{-2}\left(i\omega\mathrm{E}[\tau_a] + \frac{\omega^2}{2}\,\mathrm{E}[\tau_a^2]\right)\left(i\omega\mathrm{E}[\tau_b] + \frac{\omega^2}{2}\,\mathrm{E}[\tau_b^2]\right)}{i\omega\mathrm{E}[\tau_a] + i\omega\mathrm{E}[\tau_b] + \frac{\omega^2}{2}\,\mathrm{E}[\tau_a^2] + \frac{\omega^2}{2}\,\mathrm{E}[\tau_b^2] + \omega^2\,\mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]}\right\} \\
&\quad = \lim_{\omega\to 0}\mathrm{Re}\left\{\frac{-2\mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b] + i\omega\big(\mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b^2] + \mathrm{E}[\tau_b]\,\mathrm{E}[\tau_a^2]\big)}{2i\omega\big(\mathrm{E}[\tau_a] + \mathrm{E}[\tau_b]\big) + \omega^2\big(\mathrm{E}[\tau_a^2] + \mathrm{E}[\tau_b^2] + 2\mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]\big)}\right\} \\
&\quad = \lim_{\omega\to 0}\mathrm{Re}\left\{\frac{s + i\omega t}{i\omega u + \omega^2 v}\right\} \\
&\quad = \lim_{\omega\to 0}\mathrm{Re}\left\{\frac{s + i\omega\, t}{i\omega u + \omega^2 v} \times \frac{v + u/(i\omega)}{v + u/(i\omega)}\right\} \\
&\quad = \lim_{\omega\to 0}\mathrm{Re}\left\{\frac{tu + sv + i\omega tv + su/(i\omega)}{u^2 + \omega^2 v^2}\right\} \\
&\quad = \lim_{\omega\to 0}\frac{tu + sv}{u^2 + \omega^2\, v^2} \\
&\quad = \frac{tu + sv}{u^2} \\
&\big(\mathrm{E}[\tau_a] + \mathrm{E}[\tau_b]\big)^3\lim_{\omega\to 0} S_X(\omega) \\
&\quad = (tu + sv)/2 \\
&\quad = \big(\mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b^2] + \mathrm{E}[\tau_b]\,\mathrm{E}[\tau_a^2]\big)\,\big(\mathrm{E}[\tau_a] + \mathrm{E}[\tau_b]\big) \\
&\qquad - \mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]\,\big(\mathrm{E}[\tau_a^2] + \mathrm{E}[\tau_b^2] + 2\mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]\big)
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{E}[\tau_a]^2\,\mathrm{E}[\tau_b^2] + \mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]\,\mathrm{E}[\tau_a^2] + \mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]\,\mathrm{E}[\tau_b^2] + \mathrm{E}[\tau_b]^2\,\mathrm{E}[\tau_a^2] \\
&\qquad - \mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]\,\mathrm{E}[\tau_a^2] - \mathrm{E}[\tau_a]\,\mathrm{E}[\tau_b]\,\mathrm{E}[\tau_b^2] - 2\mathrm{E}[\tau_a]^2\,\mathrm{E}[\tau_b]^2 \\
&= \mathrm{E}[\tau_a]^2\,\mathrm{E}[\tau_b^2] + \mathrm{E}[\tau_b]^2\,\mathrm{E}[\tau_a^2] - 2\mathrm{E}[\tau_a]^2\,\mathrm{E}[\tau_b]^2 \\
&= \mathrm{E}[\tau_a]^2\,\mathrm{Var}[\tau_b] + \mathrm{E}[\tau_b]^2\,\mathrm{Var}[\tau_a], \qquad \text{(A.80)}
\end{aligned}$$

which accords with Eq. (8.6).

A.5.3 Spectrum under extreme dwell-time asymmetry

Consider an alternating renewal process $X(t)$ for which the times τ_b spent in the state $X(t) = b$ greatly exceed the times τ_a spent in state $X(t) = a$. More formally, given a randomly selected pair of inter-transition intervals τ_a and τ_b, we assume that the relation $\Pr\{\tau_a \ll \tau_b\} \approx 1$ holds. In this case, the sum of the dwell times, $\tau_a + \tau_b$, will have marginal statistics that are nearly the same as those of τ_b, and the process will closely resemble a filtered version of a renewal point process $dN(t)$ constructed solely from the longer intervals τ_b.

Linear systems theory (Papoulis, 1991) leads directly to the spectrum. Citing Eq. (9.35) in an approximate sense, we have

$$S_X(f) \approx \mathrm{E}\Big[\,|H(f)|^2\Big]\, S_N(f), \qquad \text{(A.81)}$$

where $S_N(f)$ denotes the spectrum of the renewal point process $dN(t)$ constructed from τ_b, $H(f)$ represents the Fourier transform of the filter (whose form we will establish shortly), and $S_X(f)$ is the spectrum of the resulting alternating renewal process. This approximation remains valid for all nonzero frequencies, but it does not hold for $f = 0$; at this frequency the difference between point processes and real-valued processes requires us to invoke other methods.

Specifying the impulse response function $h(t)$ to be a rectangular filter of (random) duration τ_a, we have

$$\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} \exp(-i2\pi f t)\, h(t)\, dt \\
&= \int_0^{\tau a} \exp(-i2\pi f t)\, dt \\
&= \big[1 - \exp(-i2\pi f \tau_a)\big]\big/(i2\pi f) \\
&= \exp(-i\pi f \tau_a)\, \sin(\pi f \tau_a)\big/(\pi f) \\
|H(f)|^2 &= \sin^2(\pi f \tau_a)\big/(\pi f)^2 \\
&\approx \tau_a^2 \\
\mathrm{E}\Big[\,|H(f)|^2\Big] &\approx \mathrm{E}\big[\tau_a^2\big], \qquad \text{(A.82)}
\end{aligned}$$

where we have made use of the approximation $\sin(x) \approx x$ for small arguments x, which is valid in the domain $f\tau_a \ll 1$. Finally, substituting Eqs. (4.16) and (A.82) into Eq. (A.81) leads to

$$S_X(f) \approx \frac{\mathrm{E}\left[\tau_a^2\right]}{\mathrm{E}[\tau_b]} \,\mathrm{Re}\left\{ \frac{1+\phi_{\tau b}(2\pi f)}{1-\phi_{\tau b}(2\pi f)} \right\}. \tag{A.83}$$

For the contribution at $f=0$, we note that Eq. (8.5) contains a term $\mathrm{E}[X]\,\delta(f)$; in the limit considered here, $\mathrm{E}[X]$ approaches $\mathrm{E}[\tau_a] \big/ \mathrm{E}[\tau_b]$, which agrees with Eq. (8.9).

A.6 FRACTAL SHOT NOISE

A.6.1 Infinite-area tail

If the impulse response function $h(t)$ has infinite area in its tail, the resulting shot noise process is degenerate (Lowen & Teich, 1990, Appendix A). Such an impulse response function has the property

$$\int_c^\infty h(t)\,dt = \infty \tag{A.84}$$

for any finite real number c. We rewrite Eq. (9.3), considering deterministic K for simplicity, to obtain

$$\begin{aligned}
\ln\left[\phi_X(\omega)\right] &= -\mu \int_{-\infty}^{\infty} \{1-\exp[-i\omega h(t)]\}\; dt \\
&= -\mu \int_{-\infty}^{c} \{1-\exp[-i\omega h(t)]\}\; dt \\
&\quad -\mu \int_{c}^{\infty} \{1-\exp[-i\omega h(t)]\}\; dt \\
&= -\mu f(c) - \mu \int_{c}^{\infty} \{1-\exp[-i\omega h(t)]\}\; dt \\
&\approx -\mu f(c) - \mu \int_{c}^{\infty} \{1-[1-i\omega h(t)]\}\; dt \qquad \text{(A.85)} \\
&= -\mu f(c) - i\omega\mu \int_{c}^{\infty} h(t)\,dt \\
&= -\mu f(c) - i\omega\mu\infty, \qquad \text{(A.86)}
\end{aligned}$$

where $f(c)$ denotes the value of the integral below c. We choose the value of c to be sufficiently large so that the argument of the exponential lies close to zero, permitting the approximation $\exp(-x) \approx 1-x$ in Eq. (A.85).

Of the two terms in Eq. (A.86), the second has infinite absolute value, indicating an infinite shot-noise process amplitude X, unless the first term cancels the second. However, the integrands in Eqs. (A.85) and (A.86) include a unity term, so $f(c)$ must contain a real component of comparable magnitude to its imaginary component. If the imaginary components cancel, then a real component of infinite magnitude must also exist, again leading to an infinite shot-noise process. In any case, therefore, an impulse-response function with an infinite-area tail leads to a shot-noise process X with infinite amplitude. A stochastic amplitude K does not affect this conclusion.

Even a normalized version of this impulse response function leads to a degenerate shot noise process (Lowen & Teich, 1991, Appendix D). Consider the fixed-area family defined by

$$h_B(t) \equiv \begin{cases} \dfrac{\mathsf{a}\, h(t)}{\int_{-\infty}^{B} h(u)\, du} & t < B \\ 0 & \text{otherwise,} \end{cases} \tag{A.87}$$

where the quantity a denotes a specified area as in Eq. (10.12). By construction, $h_B(t)$ has total area a. We again rewrite Eq. (9.3) for deterministic K, yielding

$$\begin{aligned} \ln\bigl[\phi_X(\omega)\bigr] &= -\mu \int_{-\infty}^{\infty} \{1 - \exp[-i\omega h_B(t)]\}\; dt \\ &= -\mu \int_{-\infty}^{B} \left\{1 - \exp\left[-\frac{i\omega\, \mathsf{a}\, h(t)}{\int_{-\infty}^{B} h(s)\, ds}\right]\right\} dt \\ &\approx -\mu \int_{-\infty}^{B} \left\{1 - \left[1 - \frac{i\omega\, \mathsf{a}\, h(t)}{\int_{-\infty}^{B} h(s)\, ds}\right]\right\} dt \qquad \text{(A.88)} \\ &= -\mu\, \frac{i\omega\, \mathsf{a} \int_{-\infty}^{B} h(t)\, dt}{\int_{-\infty}^{B} h(s)\, ds} \\ &= -i\omega\mu\mathsf{a}. \qquad \text{(A.89)} \end{aligned}$$

As B approaches ∞, the argument of the exponential decreases without limit, enabling the approximation $\exp(-x) \approx 1 - x$ to be used in Eq. (A.88). Equation (A.89) comprises a power series in ω, with terms of the power series identified with the cumulants of the shot-noise amplitude X (see Sec. 9.2). The lack of a second-order term (or indeed of any higher-order terms) indicates that the variance of the process assumes a value of zero; the amplitude remains fixed at the constant value $\mu\mathsf{a}$. We conclude that a normalized impulse response function with an infinite-area tail converges to a constant value, and thus leads to a degenerate process.

A.6.2 Approach to stable form

For fractal shot noise with $\beta > 1$, $A = 0$, and $B < \infty$, the amplitude of the shot-noise process X is not a stable random variable, but it does approach one as the rate μ of the driving Poisson process approaches infinity (Lowen & Teich, 1990, Appendix B).

To demonstrate this, we employ l'Hôpital's rule to find the limit $A \to 0$ in Eq. (9.5), which yields

$$\begin{aligned} \ln\bigl[\phi_X(\omega)\bigr] &= -\mu B\bigl[1 - \exp\bigl(-i\omega K B^{-\beta}\bigr)\bigr] \\ &\quad - \mu (i\omega K)^{1/\beta}\, \Gamma\bigl(1 - 1/\beta,\, i\omega K B^{-\beta}\bigr)\,. \end{aligned} \tag{A.90}$$

As B approaches ∞, we obtain

$$\begin{aligned}\ln\big[\phi_X(\omega)\big] &\approx -\mu B\big[1-\big(1-i\omega K B^{-\beta}\big)\big] \\ &\quad\; -\mu(i\omega K)^{1/\beta}\,\Gamma(1-1/\beta) \\ &\approx -i\mu\omega K B^{1-\beta} - \mu(i\omega K)^{1/\beta}\,\Gamma(1-1/\beta) \\ &\approx -\mu(i\omega K)^{1/\beta}\,\Gamma(1-1/\beta)\,, \end{aligned} \tag{A.91}$$

thereby validating the equivalent result shown in Eq. (9.13), but for deterministic K only.

For random K, we turn to Lebesgue measure theory (Lowen & Teich, 1990). Suppose that a stochastic impulse response function $h_1(K_1,t)$ and a deterministic impulse response function $h_2(t)$ obey the relation

$$\mathrm{E}\big[\mathcal{L}\{t : h_1(K_1,t) > x\}\big] = \mathcal{L}\{t : h_2(t) > x\} \tag{A.92}$$

for all amplitudes x, where $\mathcal{L}$ denotes the Lebesgue set measure. Shot-noise processes constructed from these two impulse response functions will then have identical first-order statistics (Gilbert & Pollak, 1960). For fractal shot noise, in particular, any stochastic impulse response function $h_1(K_1,t)$ satisfying

$$\mathrm{E}\big[\mathcal{L}\{t : h_1(K_1,t) > x\}\big] = \begin{cases} \infty & x < 0 \\ B - A & 0 \le x \le K_2\,B^{-\beta} \\ (x/K_2)^{-1/\beta} - A & K_2\,B^{-\beta} < x < K_2\,A^{-\beta} \\ 0 & x \ge K_2\,A^{-\beta} \end{cases} \tag{A.93}$$

for some β, A, B, and K_2, has identical first-order statistics as the deterministic impulse response function (Lowen & Teich, 1990)

$$h_2(K_2,t) = \begin{cases} K_2\,t^{-\beta} & A \le t < B \\ 0 & \text{otherwise.} \end{cases} \tag{A.94}$$

In general, finding a nontrivial ensemble of impulse response functions for which the equivalent impulse response function follows the form of Eq. (A.94) proves difficult. However, for the particular case $A = 0$ and $B = \infty$ we find

$$\begin{aligned}\mathrm{E}\big[\mathcal{L}\{t : h_1(K_1,t) > x\}\big] &= \mathrm{E}\big[\mathcal{L}\{t : K_1\,t^{-\beta} > x\}\big] \\ &= \mathrm{E}\Big[\mathcal{L}\{t : t < K_1^{1/\beta}\,x^{-1/\beta}\}\Big] \\ &= \mathrm{E}\Big[K_1^{1/\beta}\Big]\,x^{-1/\beta} \end{aligned} \tag{A.95}$$

for all amplitudes x. For the deterministic power-law impulse response function, we have

$$\begin{aligned}\mathcal{L}\{t : h_2(K_2,t) > x\} &= \mathcal{L}\{t : K_2\,t^{-\beta} > x\} \\ &= \mathcal{L}\{t : t < K_2^{1/\beta}\,x^{-1/\beta}\} \\ &= K_2^{1/\beta}\,x^{-1/\beta}, \end{aligned} \tag{A.96}$$

again for all amplitudes x. Thus, the stochastic ensemble of impulse response functions in Eq. (A.95) and the deterministic impulse response function in Eq. (A.96) exhibit identical first-order amplitude statistics, provided

$$\mathrm{E}\Big[K_1^{1/\beta}\Big] = K_2^{1/\beta}, \tag{A.97}$$

so that

$$K_2 \equiv \mathrm{E}^{\beta}\Big[K_1^{1/\beta}\Big]\,. \tag{A.98}$$

For $A > 0$ or $B < \infty$, Eqs. (A.95) and (A.96) are no longer in accord for all x, so that the equivalent impulse response function does not have the form of Eq. (A.96). But for the case of interest, we indeed have $A = 0$ and $B = \infty$, permitting $\mathrm{E}\big[K^{1/\beta}\big]^{\beta}$ to be used in place of deterministic K in all first-order statistics, including Eq. (9.13).

For finite B, the process still approaches a stable form for large values of μ. However, merely increasing μ leads to a degenerate characteristic function; normalization becomes necessary. To demonstrate convergence to a particular form, we therefore consider the limit $\mu \to \infty$, $K \to 0$, with the dimensionless product $\mu^{\beta}K$ fixed at a value of ω_0^{-1}. Considering the above limit, Eq. (A.90) becomes

$$\begin{aligned}
\ln\big[\phi_X(\omega)\big] \quad &\to \quad -\mu B\big[1 - \exp\big(-i\omega\,\omega_0^{-1}\,\mu^{-\beta}\,B^{-\beta}\big)\big] \\
&\qquad - \mu(i\omega\,\omega_0^{-1}\,\mu^{-\beta})^{1/\beta}\,\Gamma\big(1 - 1/\beta,\, i\omega\,\omega_0^{-1}\,\mu^{-\beta}\,B^{-\beta}\big) \\
&= \quad -x\,\big[1 - \exp\big(-ix^{-\beta}\,\omega/\omega_0\big)\big] \\
&\qquad - (i\omega/\omega_0)^{1/\beta}\,\Gamma\big(1 - 1/\beta,\, ix^{-\beta}\,\omega/\omega_0\big)\,,
\end{aligned} \tag{A.99}$$

where we define $x \equiv \mu B$. As μ increases with B fixed, the quantity x increases commensurately. The argument of the exponential in Eq. (A.99) then decreases, permitting the following simplification to be applied:

$$\begin{aligned}
\ln\big[\phi_X(\omega)\big] \quad &\to \quad -x\,\big[1 - \big(1 - ix^{-\beta}\,\omega/\omega_0\big)\big] \\
&\qquad - (i\omega/\omega_0)^{1/\beta}\,\Gamma\big(1 - 1/\beta,\, ix^{-\beta}\,\omega/\omega_0\big) \\
&= \quad -ix^{1-\beta}\,\omega/\omega_0 - (i\omega/\omega_0)^{1/\beta}\,\Gamma\big(1 - 1/\beta,\, ix^{-\beta}\,\omega/\omega_0\big) \\
&\to \quad 0 - (i\omega/\omega_0)^{1/\beta}\,\Gamma(1 - 1/\beta)\,.
\end{aligned} \tag{A.100}$$

The result is in the precise form of a stable characteristic function, as defined in Eq. (9.14).

A.6.3 Autocorrelation

We proceed to provide expressions for the shot-noise autocorrelation for specific parameter ranges.

- For $\beta = \frac{1}{2}$ and $0 \leq |t| < B - A$, we have

$$R_h(t) \quad = \quad \mathrm{E}\big[K^2\big] \int_A^{B-|t|} (s^2 + |t|\,s)^{-1/2}\,ds$$

$$
\begin{aligned}
&= 2\mathrm{E}\big[K^2\big]\ \ln\Big[s^{1/2} + (s+|t|)^{1/2}\Big]_A^{B-|t|} \\
&= 2\mathrm{E}\big[K^2\big]\ \ln\left[\frac{B^{1/2} + (B-|t|)^{1/2}}{A^{1/2} + (A+|t|)^{1/2}}\right], \qquad \text{(A.101)}
\end{aligned}
$$

finite if $B < \infty$ and either $A > 0$ or $t \neq 0$.

- For $\beta = 1$ and $t = 0$,

$$
\begin{aligned}
R_h(t) &= \mathrm{E}\big[K^2\big] \int_A^B (s^2)^{-1}\, ds \\
&= \mathrm{E}\big[K^2\big]\ \big[A^{-1} - B^{-1}\big] \qquad \text{(A.102)}
\end{aligned}
$$

is finite when $A > 0$. For $\beta = 1$ and $0 < |t| < B - A$, another logarithmic form emerges,

$$
\begin{aligned}
R_h(t) &= \mathrm{E}\big[K^2\big] \int_A^{B-|t|} (s^2 + |t|\, s)^{-1}\, ds \\
&= \mathrm{E}\big[K^2\big]\ |t|^{-1}\ \ln\left[\frac{|t|}{s+|t|}\right]_A^{B-|t|} \\
&= \mathrm{E}\big[K^2\big]\ |t|^{-1}\ \ln\Big[(1-|t|/B)\,(1+|t|/A)\Big], \qquad \text{(A.103)}
\end{aligned}
$$

which is finite if $A > 0$.

- For $\beta = 2$ and $t = 0$ we obtain

$$
\begin{aligned}
R_h(t) &= \mathrm{E}\big[K^2\big] \int_A^B (s^2)^{-2}\, ds \\
&= \tfrac{1}{3}\, \mathrm{E}\big[K^2\big]\ \big[A^{-3} - B^{-3}\big], \qquad \text{(A.104)}
\end{aligned}
$$

finite for $A > 0$. For $\beta = 2$ and $0 < |t| < B - A$, we have

$$
\begin{aligned}
R_h(t) &= \mathrm{E}\big[K^2\big] \int_A^{B-|t|} (s^2 + |t|\, s)^{-2}\, ds \\
&= -\mathrm{E}\big[K^2\big]\ \left[\frac{2s+|t|}{|t|^2\, s(s+|t|)} + \frac{2}{|t|^3}\ \ln\frac{s}{s+|t|}\right]_A^{B-|t|} \\
&= \mathrm{E}\big[K^2\big] \left\{\frac{2A+|t|}{|t|^2\, A(A+|t|)} - \frac{2B-|t|}{|t|^2\, B(B-|t|)}\right. \\
&\qquad \left. + 2|t|^{-3}\ \ln\Big[(1-|t|/B)\,(1+|t|/A)\Big]\right\}, \qquad \text{(A.105)}
\end{aligned}
$$

finite when $A > 0$.

- For general $\beta > 1$, $A > 0$, and $B = \infty$, a simple form for $R_h(t)$ emerges in the limit $|t| \to \infty$. We first find an upper bound for the integral:

$$\begin{aligned}\int_A^\infty \left(s^2 + |t|\, s\right)^{-\beta} ds &= \int_A^\infty s^{-\beta} \left(s + |t|\right)^{-\beta} ds \\ &< \int_A^\infty s^{-\beta} \left(|t|\right)^{-\beta} ds \\ &= \frac{A^{1-\beta}}{\beta - 1} |t|^{-\beta}. \end{aligned} \tag{A.106}$$

For the lower bound we truncate the integral at some value T,

$$\begin{aligned}\int_A^\infty \left(s^2 + |t|\, s\right)^{-\beta} ds &= \int_A^\infty s^{-\beta} (s + |t|)^{-\beta} ds \\ &> \int_A^T s^{-\beta} \left(T + |t|\right)^{-\beta} ds \\ &= \frac{A^{1-\beta} - T^{1-\beta}}{\beta - 1} \left(T + |t|\right)^{-\beta}, \end{aligned} \tag{A.107}$$

valid for any $T > A$. We choose $T = (A|t|)^{1/2}$, so that

$$\begin{aligned}\int_A^\infty \left(s^2 + |t|\, s\right)^{-\beta} ds &> \frac{A^{1-\beta} - (A|t|)^{(1-\beta)/2}}{\beta - 1} \left[(A|t|)^{1/2} + |t|\right]^{-\beta} \\ &= \frac{A^{1-\beta}}{\beta - 1} |t|^{-\beta} \left[1 + (A/|t|)^{1/2}\right]^{-\beta} \\ &\quad \times \left[1 - (A/|t|)^{(\beta-1)/2}\right]. \end{aligned} \tag{A.108}$$

Finally, combining limits yields

$$\begin{aligned}&\left[1 + (A/|t|)^{1/2}\right]^{-\beta} \left[1 - (A/|t|)^{(\beta-1)/2}\right] \\ &\quad < \frac{\int_A^\infty \left(s^2 + |t|\, s\right)^{-\beta} ds}{|t|^{-\beta} A^{1-\beta}/(\beta - 1)} < 1, \end{aligned} \tag{A.109}$$

for all t such that $|t| > A$. In the limit $|t| \to \infty$, the lower bound approaches unity, so that

$$\int_A^\infty \left(s^2 + |t|\, s\right)^{-\beta} ds \to |t|^{-\beta} A^{1-\beta}/(\beta - 1) \tag{A.110}$$

and

$$R_h(t) \to \mathrm{E}\left[K^2\right] \frac{A^{1-\beta}}{\beta - 1} |t|^{-\beta}. \tag{A.111}$$

A.7 FRACTAL-SHOT-NOISE-DRIVEN POINT PROCESSES

A.7.1 Integrals for counting statistics

The counting distribution for the fractal-shot-noise-driven Poisson process derives from a recursion relation. We consider the case for deterministic K. Reiterating Eqs. (10.4) and (10.5), we have

$$p_Z(n+1;T) = \frac{1}{n+1} \sum_{k=0}^{n} c_k \, p_Z(n-k;T), \tag{A.112}$$

and

$$c_k \equiv \frac{\mu}{k!} \int_{-\infty}^{\infty} \Big[h_T(K,t)\Big]^{k+1} \exp[-h_T(K,t)] \, dt. \tag{A.113}$$

The recursion coefficients c_k assume four different forms, depending on the value of β and the relative magnitudes of A, B, and T.

- For $\beta \neq 1$ and $B > A + T$:

$$\begin{aligned} c_k \;=\; & \frac{\mu \, K^{k+1}}{k! \, (1-\beta)^{k+1}} \\ & \times \left(\int_A^{A+T} \left[u^{1-\beta} - A^{1-\beta}\right]^{k+1} \exp\left\{ -\frac{K}{1-\beta} \left[u^{1-\beta} - A^{1-\beta}\right] \right\} du \right. \\ & + \int_A^{B-T} \left[(u+T)^{1-\beta} - u^{1-\beta}\right]^{k+1} \\ & \quad \times \exp\left\{ -\frac{K}{1-\beta} \left[(u+T)^{1-\beta} - u^{1-\beta}\right] \right\} du \\ & + \int_{B-T}^{B} \left[B^{1-\beta} - u^{1-\beta}\right]^{k+1} \\ & \quad \left. \times \exp\left\{ -\frac{K}{1-\beta} \left[B^{1-\beta} - u^{1-\beta}\right] \right\} du \right). \end{aligned} \tag{A.114}$$

- For $\beta \neq 1$ and $B \leq A + T$:

$$\begin{aligned} c_k \;=\; & \frac{\mu \, K^{k+1}}{k! \, (1-\beta)^{k+1}} \\ & \times \left(\int_A^{B} \left[u^{1-\beta} - A^{1-\beta}\right]^{k+1} \right. \\ & \quad \times \exp\left\{ -\frac{K}{1-\beta} \left[(u+T)^{1-\beta} - A^{1-\beta}\right] \right\} du \\ & + \int_A^{B} \left[B^{1-\beta} - u^{1-\beta}\right]^{k+1} \exp\left\{ -\frac{K}{1-\beta} \left[B^{1-\beta} - u^{1-\beta}\right] \right\} du \end{aligned}$$

$$+ (T + A - B)\,\left[B^{1-\beta} - A^{1-\beta}\right]^{k+1} \times \exp\left\{-\frac{K}{1-\beta}\left[B^{1-\beta} - A^{1-\beta}\right]\right\}\Bigg). \tag{A.115}$$

- For $\beta = 1$ and $B > A + T$:

$$\begin{aligned} c_k &= \frac{\mu K^{k+1}}{k!}\left\{\int_A^{B-T}\left[\ln\left(\frac{u+T}{u}\right)\right]^{k+1}\left(\frac{u}{u+T}\right)^K du \right. \\ &\quad + \int_A^{A+T}\left[\ln\left(\frac{u}{A}\right)\right]^{k+1}\left(\frac{A}{u}\right)^K du \\ &\quad \left. + \int_{B-T}^{B}\left[\ln\left(\frac{B}{u}\right)\right]^{k+1}\left(\frac{u}{B}\right)^K du\right\}. \end{aligned} \tag{A.116}$$

- Finally, for $\beta = 1$ and $B \le A + T$:

$$\begin{aligned} c_k &= \frac{\mu K^{k+1}}{k!}\left\{(T + A - B)\left[\ln\left(\frac{B}{A}\right)\right]^{k+1}\left(\frac{A}{B}\right)^K \right. \\ &\quad + \int_A^{B}\left[\ln\left(\frac{u}{A}\right)\right]^{k+1}\left(\frac{A}{u}\right)^K du \\ &\quad \left. + \int_{A}^{B}\left[\ln\left(\frac{B}{u}\right)\right]^{k+1}\left(\frac{u}{B}\right)^K du\right\}. \end{aligned} \tag{A.117}$$

The count moments of the fractal-shot-noise-driven Poisson process also derive from a recursion relation, but in this case we can easily consider stochastic as well as deterministic K. Reiterating Eqs. (10.8) and (10.9), we have

$$\mathrm{E}\left\{\frac{[Z(t)]!}{[Z(t) - (n+1)]!}\right\} = \sum_{k=0}^{n} b_k \binom{n}{k} \mathrm{E}\left\{\frac{[Z(t)]!}{[Z(t) - (n-k)]!}\right\}, \tag{A.118}$$

with

$$\mathrm{E}\left\{\frac{[Z(t)]!}{[Z(t)]!}\right\} \equiv 1 \qquad \text{and} \qquad b_k \equiv \mu\,\mathrm{E}\left[\int_{-\infty}^{\infty}\left[h_T(K,t)\right]^{k+1} dt\right]. \tag{A.119}$$

The recursion coefficients b_k also have four different forms, depending on β and the relative magnitudes of A, B, and T:

- For $\beta \ne 1$ and $B > A + T$:

$$b_k = \frac{\mu\,\mathrm{E}[K^{k+1}]}{k!\,(1-\beta)^{k+1}}\left\{\int_A^{B-T}\left[(u+T)^{1-\beta} - u^{1-\beta}\right]^{k+1} du\right.$$

$$+ \int_A^{A+T} \left[u^{1-\beta} - A^{1-\beta}\right]^{k+1} du$$
$$+ \int_{B-T}^{B} \left[B^{1-\beta} - u^{1-\beta}\right]^{k+1} du \Bigg\}. \quad \text{(A.120)}$$

- For $\beta \neq 1$ and $B \leq A + T$:

$$b_k = \frac{\mu \, \mathrm{E}[K^{k+1}]}{k! \, (1-\beta)^{k+1}} \Bigg\{ (T + A - B) \left[B^{1-\beta} - A^{1-\beta}\right]^{k+1} + \int_A^{B} \left[u^{1-\beta} - A^{1-\beta}\right]^{k+1} du + \int_A^{B} \left[B^{1-\beta} - u^{1-\beta}\right]^{k+1} du \Bigg\}. \quad \text{(A.121)}$$

- For $\beta = 1$ and $B > A + T$:

$$b_k = \frac{\mu \, \mathrm{E}[K^{k+1}]}{k!} \Bigg\{ \int_A^{B-T} \left[\ln\left(\frac{u+T}{u}\right)\right]^{k+1} du + \int_A^{A+T} \left[\ln\left(\frac{u}{A}\right)\right]^{k+1} du + \int_{B-T}^{B} \left[\ln\left(\frac{B}{u}\right)\right]^{k+1} du \Bigg\}. \quad \text{(A.122)}$$

- Finally, for $\beta = 1$ and $B \leq A + T$:

$$b_k = \frac{\mu \, \mathrm{E}[K^{k+1}]}{k!} \Bigg\{ (T + A - B) \left[\ln\left(\frac{B}{A}\right)\right]^{k+1} + \int_A^{B} \left[\ln\left(\frac{u}{A}\right)\right]^{k+1} du + \int_A^{B} \left[\ln\left(\frac{B}{u}\right)\right]^{k+1} du \Bigg\}. \quad \text{(A.123)}$$

A.7.2 Expressions for normalized variance

General closed-form expressions for the fractal-shot-noise-driven Poisson-process normalized variance $F(T)$ do not exist. However, in some special cases and limits one can indeed find such forms, and we present their detailed derivations below.

- For $\beta = \frac{1}{2}$ the normalized variance becomes

$$F(T) = 1 + \frac{\mathrm{E}[K^2]}{T\,\mathrm{E}[K]\big(B^{1/2} - A^{1/2}\big)} \int_0^{\Phi} (T-u) \int_A^{B-u} (t^2 + u\,t)^{-1/2}\,dt\,du, \tag{A.124}$$

where we define the upper limit of the outer integral as

$$\Phi \equiv \min(T, B-A), \tag{A.125}$$

namely the smaller of T and $B-A$. For the inner integral we have

$$\int_A^{B-u} (t^2 + u\,t)^{-1/2}\,dt = 2\left\{\ln\left[t^{1/2} + (t+u)^{1/2}\right]\right\}_A^{B-u}, \tag{A.126}$$

so that the outer integral simplifies to

$$\begin{aligned} &2\int_0^{\Phi} (T-u)\,\ln\left[1 + (1-u/B)^{1/2}\right]\,du \\ &\quad - 2\int_0^{\Phi} (T-u)\,\ln\left[1 + (1+u/A)^{1/2}\right]\,du \\ &\quad + \ln\left(\frac{B^{1/2}}{A^{1/2}}\right)\,(2T\Phi - \Phi^2). \end{aligned} \tag{A.127}$$

The remaining integrals in Eq. (A.127) follow the form

$$\begin{aligned} &2\int_0^{\Phi} (T-u)\,\ln\left[1 + (1+u/c)^{1/2}\right]du \\ &= 4\,c\,(T+c)\int_1^{(1+\Phi/c)^{1/2}} v\,\ln(1+v)\,dv \\ &\qquad - 4\,c^2 \int_1^{(1+\Phi/c)^{1/2}} v^3 \ln(1+v)\,dv \qquad \text{(A.128)} \\ &= \Phi(2T-\Phi)\,\ln\left[1 + (1+\Phi/c)^{1/2}\right] \\ &\qquad + c\left[\tfrac{2}{3}c + 2T - \tfrac{1}{3}\Phi\right](1+\Phi/c)^{1/2} \\ &\qquad - \tfrac{2}{3}c^2 - 2cT - T\Phi + \tfrac{1}{4}\Phi^2, \qquad \text{(A.129)} \end{aligned}$$

where Eq. (A.128) derives from the substitution $v \equiv (1+u/c)^{1/2}$. This yields the following expression for the normalized variance itself when $\beta = \frac{1}{2}$:

$$F(T) \quad = \quad 1 + \frac{\mathrm{E}[K^2]}{T\,\mathrm{E}[K]\left(B^{1/2} - A^{1/2}\right)}$$

$$\times \Bigg\{ \Phi(2T-\Phi)\, \ln\left[\frac{B^{1/2}+(B-\Phi)^{1/2}}{A^{1/2}+(A+\Phi)^{1/2}}\right] + \tfrac{1}{3}(\Phi-6T+2B)\,(B^2-B\Phi)^{1/2} + \tfrac{1}{3}(\Phi-6T-2A)\,(A^2+A\Phi)^{1/2} - \tfrac{2}{3}(B^2-A^2)+2T(B+A) \Bigg\}. \quad \text{(A.130)}$$

- For $\beta = 2$ the normalized variance becomes

$$F(T) = 1 + \frac{2AB\,\mathrm{E}[K^2]}{T(B-A)\,\mathrm{E}[K]} \int_0^{\Phi} (T-u) \int_A^{B-u} (t^2+u\,t)^{-2}\, dt\, du, \quad \text{(A.131)}$$

where Φ is defined as above. For the inner integral we have

$$\begin{aligned} &\int_A^{B-u} (t^2+u\,t)^{-2}\, dt \\ &= \int_A^{B-u} \frac{1}{u^3}\left[\frac{u}{t^2} + \frac{u}{(t+u)^2} - \frac{2}{t} + \frac{2}{t+u}\right] dt \\ &= u^{-3}\left[-\frac{u}{t} - \frac{u}{t+u} - 2\ln(t) + 2\ln(t+u)\right]_A^{B-u}, \end{aligned} \quad \text{(A.132)}$$

so that the outer integral simplifies to

$$\left(\frac{T}{u^2} - \frac{2}{u} - \frac{1}{A}\right) \ln(1+u/A) - \frac{T}{A\,u} + \left(\frac{T}{u^2} - \frac{2}{u} + \frac{1}{B}\right) \ln(1-u/B) + \frac{T}{B\,u}\Bigg|_{u=0}^{\Phi}. \quad \text{(A.133)}$$

The quantity in Eq. (A.133) is not defined in the limit $u \to 0$, so we use l'Hôpital's rule to obtain

$$\begin{aligned} &\lim_{u\to 0}\left[\left(\frac{T}{u^2} - \frac{2}{u} - \frac{1}{A}\right) \ln(1+u/A) - \frac{T}{A\,u}\right] \\ &= T \lim_{u\to 0} \frac{\ln(1+u/A) - u/A}{u^2\, r} \\ &\quad - 2\lim_{u\to 0} \frac{\ln(1+u/A)}{u} - \frac{1}{A}\lim_{u\to 0} \ln(1+u/A) \\ &= -\frac{T}{2A^2} - \frac{2}{A} - 0. \end{aligned} \quad \text{(A.134)}$$

Similarly, we obtain $-T/2B^2 + 2/B$ for the second pair of terms in Eq. (A.133). This yields the following expression for the normalized variance itself when

$\beta = 2$:

$$\begin{aligned} F(T) &= 1 + \frac{2AB\,\mathrm{E}[K^2]}{T(B-A)\,\mathrm{E}[K]} \left[\frac{T}{2A^2} + \frac{2}{A} - \frac{T}{A\Phi} + \frac{T}{2B^2} - \frac{2}{B} + \frac{T}{B\Phi} \right. \\ &\quad + \left(\frac{T}{\Phi^2} - \frac{2}{\Phi} - \frac{1}{A} \right) \ln(1 + \Phi/A) \\ &\quad \left. + \left(\frac{T}{\Phi^2} - \frac{2}{\Phi} + \frac{1}{B} \right) \ln(1 - \Phi/B) \right]. \end{aligned} \tag{A.135}$$

In contrast to the exact expressions for the normalized variance that are available for the two specific values of β considered above, approximate expressions can be obtained for arbitrary β in the following limits: $T \ll A$, $A \ll T \ll B$, and $T \gg B$. Rather than considering limits of the entire normalized variance expression

$$F(T) = 1 + \frac{2\mathrm{E}[K^2]}{T\,\mathrm{E}[K] \displaystyle\int_A^B t^{-\beta}\,dt} \int_0^\Phi (T-u) \int_A^{B-u} (t^2 + u\,t)^{-\beta}\,dt\,du, \tag{A.136}$$

we obtain limits for the integrals within this expression.

- For $T \ll A$, we have $\Phi = T$. By using l'Hôpital's rule twice we obtain

$$\lim_{T\to 0} \int_0^T (T-u) \int_A^{B-u} (t^2+u\,t)^{-\beta}\,dt\,du \Big/ T^2 = \frac{1}{2} \int_A^B (t^2)^{-\beta}\,dt, \tag{A.137}$$

so that for small T,

$$\begin{aligned} F(T) &\approx 1 + \frac{\mathrm{E}[K^2] \displaystyle\int_A^B t^{-2\beta}\,dt}{E[K] \displaystyle\int_A^B t^{-\beta}\,dt}\, T \\ &= 1 + \frac{\mathrm{E}[K^2]}{\mathsf{a}} \left[\int_A^B t^{-2\beta}\,dt \right] T, \end{aligned} \tag{A.138}$$

as provided in Eq. (10.15).

- For $A \ll T \ll B$, again we have $\Phi = T$, but now the limiting expression depends on β. Since in this case $A \ll B$, the integral in the denominator of the normalized-variance expression provided in Eq. (A.136) tends to a simple limit as $B/A \to \infty$:

$$\int_A^B t^{-\beta}\,dt \to \begin{cases} B^{1-\beta}/(1-\beta) & \beta < 1 \\ \ln(B/A) & \beta = 1 \\ A^{1-\beta}/(\beta - 1) & \beta > 1. \end{cases} \tag{A.139}$$

The double integral in Eq. (A.136), henceforth denoted Ω, has a more complex form; we consider in turn five expressions for different ranges of β.

1. For $0 < \beta < \frac{1}{2}$, we define $x \equiv A/T$ and $y \equiv B/T$, so that

$$\Omega = T^{3-2\beta} \int_0^1 (1-u) \int_x^{y-u} (t^2 + u\,t)^{-\beta}\, dt\, du. \qquad \text{(A.140)}$$

Setting $x = 0$ and using l'Hôpital's rule leads to

$$\lim_{y\to\infty} \int_0^1 (1-u) \int_0^{y-u} (t^2 + u\,t)^{-\beta}\, dt\, du \Big/ y^{1-2\beta} = [2(1-2\beta)]^{-1}\,, \qquad \text{(A.141)}$$

so that the normalized variance becomes

$$F(T) \approx 1 + \frac{\mathrm{E}[K^2]\,(1-\beta)}{\mathrm{E}[K]\,(1-2\beta)}\, B^{-\beta}\, T, \qquad \text{(A.142)}$$

as provided in Eq. (10.16).

2. For $\beta = \frac{1}{2}$, we again set $x = 0$ and use l'Hôpital's rule to obtain

$$\lim_{y\to\infty} \int_0^1 (1-u) \int_0^{y-u} (t^2 + u\,t)^{-1/2}\, dt\, du \Big/ \ln(y) = \frac{1}{2}, \qquad \text{(A.143)}$$

so that

$$F(T) \approx 1 + \frac{\mathrm{E}[K^2]}{\mathrm{E}[K]}\, \frac{1}{2} B^{-1/2}\, [\ln(B/T)]\, T. \qquad \text{(A.144)}$$

3. For $\frac{1}{2} < \beta < 1$, we consider the limits in which both $x \to 0$ and $y \to \infty$. Here the integral in the numerator becomes

$$\begin{aligned} \Omega &= T^{3-2\beta} \int_0^1 (1-u) \int_0^\infty (t^2 + u\,t)^{-\beta}\, dt\, du \\ &= T^{3-2\beta} \int_0^1 (1-u) \int_0^\infty (u^2x^2 + u^2x)^{-\beta}\, u\, dx\, du \qquad \text{(A.145)} \\ &= T^{3-2\beta} \int_0^1 (1-u)\, u^{1-2\beta}\, du \int_0^\infty (x^2 + x)^{-\beta}\, dx \\ &= \frac{T^{3-2\beta}}{2(1-\beta)\,(3-2\beta)}\, \frac{\Gamma(1-\beta)\,\Gamma(2\beta-1)}{\Gamma(\beta)}\,, \qquad \text{(A.146)} \end{aligned}$$

where Eq. (A.145) derives from the substitution $x \equiv t/u$ in the inner integral. The normalized variance then becomes

$$F(T) \approx 1 + \frac{\mathrm{E}[K^2]\,\Gamma(1-\beta)\,\Gamma(2\beta-1)}{E[K]\,(3-2\beta)\,\Gamma(\beta)}\, B^{\beta-1}\, T^{2(1-\beta)}, \qquad \text{(A.147)}$$

which concurs with Eq. (10.16) when the definition $\alpha \equiv 2(1-\beta)$ is used.

4. For $\beta = 1$ we define $x \equiv T/A$ and $y \equiv T/B$ to obtain

$$\begin{aligned}\Omega &= \int_0^T (T-u) \int_A^{B-u} (t^2 + u\,t)^{-1}\, dt\, du \\ &= T \int_0^1 \frac{1-u}{u}\ \ln(1 - y\,u)\ du \\ &\quad + T \int_0^1 \frac{1-u}{u}\ \ln(1 + x\,u)\ du. \qquad \text{(A.148)}\end{aligned}$$

The first term in Eq. (A.148) approaches zero as $y \to 0$ since

$$\begin{aligned}0 &> T \int_0^1 \frac{1-u}{u}\ \ln(1 - y\,u)\, du \\ &> T \int_0^1 \frac{1-u}{u}\ u\ \ln(1-y)\, du \\ &= \tfrac{1}{2}\, T\ \ln(1-y), \qquad \text{(A.149)}\end{aligned}$$

and $\ln(1-y) \to 0$ as $y \to 0$. For the second term in Eq. (A.148), two applications of l'Hôpital's rule and some simplification yield

$$\lim_{x\to\infty} \int_0^1 \frac{1-u}{u}\ \ln(1 + x\,u)\ du \Big/ \ln^2(x) = 1. \qquad \text{(A.150)}$$

The normalized variance therefore becomes

$$F(T) \approx 1 + \frac{2\mathrm{E}[K^2]}{\mathrm{E}[K]}\ \frac{\ln^2(T/A)}{\ln(B/A)}. \qquad \text{(A.151)}$$

5. For $\beta > 1$, we define $x \equiv T/A$ and $y \equiv B/T$ to obtain

$$\Omega = A^{3-2\beta} \int_0^x (x-u) \int_1^{xy-u} (t^2 + u\,t)^{-\beta}\, dt\, du. \qquad \text{(A.152)}$$

Setting $y > 1$ and using l'Hôpital's rule yields

$$\lim_{x\to\infty} \int_0^x (x-u) \int_1^{xy-u} (t^2 + u\,t)^{-\beta}\, dt\, du \Big/ x = \left[2(\beta-1)^2\right]^{-1}, \qquad \text{(A.153)}$$

whereupon the normalized variance becomes

$$F(T) \approx 1 + \frac{\mathrm{E}[K^2]\, A^{1-\beta}}{\mathrm{E}[K]\,(\beta-1)} \approx 1 + \frac{\mathrm{E}[K^2]}{\mathrm{E}^2[K]}\ \mathsf{a}. \qquad \text{(A.154)}$$

- Finally, in the third region, where $T \gg B$, we have $\Phi = B - A$. Using the substitution $v \equiv t + u$ and interchanging the order of integration in the numerator of Eq. (A.136) yields

$$F(T) = 1 + \frac{2\mathrm{E}[K^2]}{\mathrm{E}[K]} \int_A^B t^{-\beta} \int_t^B \left(1 + \frac{t-v}{T}\right) v^{-\beta}\, dv\, dt \Big/ \int_A^B t^{-\beta}\, dt. \qquad \text{(A.155)}$$

In the limit $T \gg B$, the $(t - v)$ term in the numerator vanishes so that

$$\begin{aligned} F(T) &\approx 1 + \frac{2\mathrm{E}[K^2]}{\mathrm{E}[K]} \int_A^B t^{-\beta} \int_t^B v^{-\beta}\, dv\, dt \bigg/ \int_A^B t^{-\beta}\, dt \\ &= 1 + \frac{2\mathrm{E}[K^2]}{\mathrm{E}[K]} \frac{1}{2} \left[\int_A^B t^{-\beta}\, dt \right]^2 \bigg/ \int_A^B t^{-\beta}\, dt \\ &= 1 + \frac{\mathrm{E}[K^2]}{\mathrm{E}[K]} \int_A^B t^{-\beta}\, dt \\ &= 1 + \frac{\mathrm{E}[K^2]}{\mathrm{E}^2[K]}\, \mathsf{a}, \end{aligned} \tag{A.156}$$

in agreement with Eq. (10.17).

A.7.3 Expressions for normalized Haar-wavelet variance

Equation (3.41) provides a relation that permits us to obtain the normalized Haar-wavelet variance $A(T)$ directly from the normalized variance $F(T)$. This direct route is suitable for all forms of $F(T)$ except those in which its leading term is linear in T [see Eqs. (5.37)–(5.39)]. This latter condition arises for $T \ll A$, and for $A \ll T \ll B$ with $\beta < \frac{1}{2}$. In these two cases, we obtain $A(T)$ using other methods, as described below.

- For $T \ll A$, we form three derivatives of the double integral in Eq. (A.136), denoted Ω, which yields

$$\begin{aligned} \Omega &= \int_0^T (T-u) \int_A^{B-u} (t^2 + u\,t)^{-\beta}\, dt\, du \\ \frac{d\Omega}{dT} &= \int_0^T \int_A^{B-u} (t^2 + u\,t)^{-\beta}\, dt\, du \\ \frac{d^2\Omega}{dT^2} &= \int_A^{B-T} (t^2 + Tt)^{-\beta}\, dt \\ \frac{d^3\Omega}{dT^3} &= -\left[B(B-T)\right]^{-\beta} - \beta \int_A^{B-T} t^{-\beta}(t+T)^{-(1+\beta)}\, dt \\ \left.\frac{d^3\Omega}{dT^3}\right|_{T=0} &= -B^{-2\beta} - \beta \int_A^B t^{-(1+2\beta)}\, dt \\ &= -\tfrac{1}{2}\left[A^{-2\beta} + B^{-2\beta}\right] \\ F(T) &\approx 1 + \frac{\mathrm{E}\left[K^2\right]}{\mathsf{a}} \int_A^B t^{-2\beta}\, dt\, T - \frac{\mathrm{E}\left[K^2\right]}{6\mathsf{a}} \left[A^{-2\beta} + B^{-2\beta}\right] T^2 \\ A(T) &\approx 1 + \frac{\mathrm{E}\left[K^2\right]}{3\mathsf{a}} \left[A^{-2\beta} + B^{-2\beta}\right] T^2. \end{aligned} \tag{A.157}$$

- For $A \ll T \ll B$ and $\beta < \frac{1}{2}$, we make use of results from Eq. (A.157) to obtain

$$\begin{aligned}
\frac{d^3\Omega}{dT^3} &= -\big[B(B-T)\big]^{-\beta} \\
&\quad - \beta \int_A^{B-T} t^{-\beta}\,(t+T)^{-(1+\beta)}\,dt \\
T^{2\beta}\,\frac{d^3\Omega}{dT^3} &= -\left[\frac{B}{T}\left(\frac{B}{T}-1\right)\right]^{-\beta} \\
&\quad - \beta \int_{A/T}^{B/T-1} x^{-\beta}\,(x+1)^{-(1+\beta)}\,dx \\
\lim_{\substack{A/T\to 0\\ B/T\to\infty}} T^{2\beta}\,\frac{d^3\Omega}{dT^3} &= -\beta \int_0^{\infty} x^{-\beta}\,(x+1)^{-(1+\beta)}\,dx \\
&= -\beta\,\frac{\Gamma(1-\beta)\,\Gamma(2\beta)}{\Gamma(1+\beta)}. \qquad \text{(A.158)}
\end{aligned}$$

Finally, then, we obtain

$$\begin{aligned}
F(T) &\approx 1 + \frac{\mathrm{E}\big[K^2\big]}{\mathsf{a}} \int_A^B t^{-2\beta}\,dt\,T - \frac{2\mathrm{E}\big[K^2\big]\,(1-\beta)}{\mathrm{E}[K]\,B^{1-\beta}\,T} \\
&\quad \times \frac{T^{3-2\beta}}{(1-2\beta)\,(2-2\beta)\,(3-2\beta)}\,\beta\,\frac{\Gamma(1-\beta)\,\Gamma(2\beta)}{\Gamma(1+\beta)} \\
&= 1 + \frac{\mathrm{E}\big[K^2\big]}{\mathsf{a}} \int_A^B t^{-2\beta}\,dt\,T \\
&\quad - \frac{\mathrm{E}\big[K^2\big]}{\mathrm{E}[K]}\,\frac{\beta\,\Gamma(1-\beta)\,\Gamma(2\beta)}{(1-2\beta)\,(3-2\beta)\,\Gamma(1+\beta)}\,\frac{T^{2-2\beta}}{B^{1-\beta}} \\
&= 1 + \frac{\mathrm{E}\big[K^2\big]}{\mathsf{a}} \int_A^B t^{-2\beta}\,dt\,T \\
&\quad - \frac{\mathrm{E}\big[K^2\big]}{\mathrm{E}[K]}\,\frac{\beta\,\Gamma(1-\beta)\,(2\beta-1)\,\Gamma(2\beta-1)}{(1-2\beta)\,(3-2\beta)\beta\,\Gamma(\beta)}\,\frac{T^{2-2\beta}}{B^{1-\beta}} \\
&= 1 + \frac{\mathrm{E}\big[K^2\big]}{\mathsf{a}} \int_A^B t^{\alpha-2}\,dt\,T \\
&\quad + \frac{\mathrm{E}\big[K^2\big]}{\mathrm{E}[K]}\,\frac{\Gamma(\alpha/2)\,\Gamma(1-\alpha)}{(1+\alpha)\,\Gamma(1-\alpha/2)}\,\frac{T^{\alpha}}{B^{\alpha/2}} \\
A(T) &\approx 1 + (2-2^{\alpha})\,\frac{\mathrm{E}\big[K^2\big]}{\mathrm{E}[K]}\,\frac{\Gamma(\alpha/2)\,\Gamma(1-\alpha)}{(1+\alpha)\,\Gamma(1-\alpha/2)}\,\frac{T^{\alpha}}{B^{\alpha/2}} \\
&= 1 + \frac{\mathrm{E}\big[K^2\big]}{\mathrm{E}[K]}\,\frac{(2^{\alpha}-2)\,\Gamma(\alpha/2)\,\Gamma(2-\alpha)}{\big(\alpha^2-1\big)\,\Gamma(1-\alpha/2)}\,\frac{T^{\alpha}}{B^{\alpha/2}}, \qquad \text{(A.159)}
\end{aligned}$$

in accordance with Eq. (10.19). This result applies for all $\beta < 1$.

A.7.4 Integrals for time statistics

Calculation of the forward-recurrence-time and interevent-interval statistics begins with $p_Z(0;T)$, the probability that there are zero events in an interval of duration T chosen independently of the process, and proceeds to its first two derivatives.

From Eq. (10.3), in the special case of deterministic K, we have

$$p_Z(0;T) = \exp\left(\mu \int_{-\infty}^{\infty} \{\exp[-h_T(u)] - 1\}\, du\right) = \exp[\mu f(T)]\,, \qquad \text{(A.160)}$$

where the quantity $f(T)$, implicitly defined in Eq. (A.160), serves to simplify the notation. In accordance with the results provided in Sec. 3.3.1, the forward-recurrence-time probability density then becomes

$$p_\vartheta(t) = -\frac{d}{dT}\Big[p_Z(0;T)\Big]_{T=t} = -\mu p_Z(0;t)\,\frac{df(t)}{dt} \qquad \text{(A.161)}$$

while a second derivative yields the interevent-interval density:

$$\begin{aligned} p_\tau(t) &= -\frac{1}{\mathrm{E}[X]}\,\frac{d^2}{dT^2}\Big[p_Z(0;T)\Big]_{T=t} \\ &= \frac{1}{\mathsf{a}}\,p_Z(0;t)\left\{\mu\left[\frac{df(t)}{dt}\right]^2 + \frac{d^2 f(t)}{dt^2}\right\}. \end{aligned} \qquad \text{(A.162)}$$

Thus, $p_Z(0;T)$, $p_\vartheta(t)$, and $p_\tau(t)$ depend, in turn, on $f(t)$ and its first two derivatives, which we calculate below.

The function $f(t)$ assumes four different forms, depending on the value of β and the relative magnitudes of A, B, and t.

- For $\beta \neq 1$ and $B > A + t$:

$$\begin{aligned} f(t) &= \int_A^{A+t}\left(\exp\left\{-\frac{K}{1-\beta}\left[u^{1-\beta} - A^{1-\beta}\right]\right\} - 1\right) du \\ &\quad + \int_A^{B-t}\left(\exp\left\{-\frac{K}{1-\beta}\left[(u+t)^{1-\beta} - u^{1-\beta}\right]\right\} - 1\right) du \\ &\quad + \int_{B-t}^{B}\left(\exp\left\{-\frac{K}{1-\beta}\left[B^{1-\beta} - u^{1-\beta}\right]\right\} - 1\right) du \\ -\frac{df(t)}{dt} &= K\int_A^{B-t}(u+t)^{-\beta}\,\exp\left\{-\frac{K}{1-\beta}\left[(u+t)^{1-\beta} - u^{1-\beta}\right]\right\} du \\ &\quad + \left(1 - \exp\left\{-\frac{K}{1-\beta}\left[(A+t)^{1-\beta} - A^{1-\beta}\right]\right\}\right) \\ \frac{d^2 f(t)}{dt^2} &= KB^{-\beta}\exp\left\{-\frac{K}{1-\beta}\left[B^{1-\beta} - (B-t)^{1-\beta}\right]\right\} \\ &\quad - K(A+t)^{-\beta}\,\exp\left\{-\frac{K}{1-\beta}\left[(A+t)^{1-\beta} - A^{1-\beta}\right]\right\} \end{aligned}$$

$$+ K \int_{A+t}^{B} \left[\beta\, u^{-\beta-1} + K u^{-2\beta}\right] \times \exp\left\{-\frac{K}{1-\beta}\left[u^{1-\beta} - (u-t)^{1-\beta}\right]\right\} du. \qquad \text{(A.163)}$$

- For $\beta \neq 1$ and $B \leq A + t$:

$$\begin{aligned} f(t) &= \int_A^B \left(\exp\left\{-\frac{K}{1-\beta}\left[u^{1-\beta} - A^{1-\beta}\right]\right\} - 1\right) du \\ &\quad + \int_A^B \left(\exp\left\{-\frac{K}{1-\beta}\left[B^{1-\beta} - u^{1-\beta}\right]\right\} - 1\right) du \\ &\quad + (t + A - B)\left(e^{-\mathsf{a}} - 1\right) \\ -\frac{df(t)}{dt} &= 1 - e^{-\mathsf{a}} \\ \frac{d^2 f(t)}{dt^2} &= 0. \end{aligned} \qquad \text{(A.164)}$$

- For $\beta = 1$ and $B > A + t$:

$$\begin{aligned} f(t) &= \int_A^{B-t} \left[\left(\frac{u}{u+t}\right)^K - 1\right] du + \int_A^{A+t} \left[\left(\frac{A}{u}\right)^K - 1\right] du \\ &\quad + \int_{B-t}^{B} \left[\left(\frac{u}{B}\right)^K - 1\right] du \\ -\frac{df(t)}{dt} &= K \int_A^{B-t} \frac{u^K}{(u+t)^{K+1}}\, du - \left[\left(\frac{A}{A+t}\right)^K - 1\right] \\ \frac{d^2 f(t)}{dt^2} &= K\frac{(B-t)^K}{B^{K+1}} - K\frac{A^K}{(A+t)^{K+1}} \\ &\quad - K(K+1)\int_A^{B-t} \frac{u^K}{(u+t)^{K+2}}\, du. \end{aligned} \qquad \text{(A.165)}$$

- Finally, for $\beta = 1$ and $B < A + t$:

$$\begin{aligned} f(t) &= \int_A^B \left[\left(\frac{A}{u}\right)^K - 1\right] du + \int_A^B \left[\left(\frac{u}{B}\right)^K - 1\right] du \\ &\quad + (t + A - B)\left[\left(\frac{A}{B}\right)^K - 1\right] \\ -\frac{df(t)}{dt} &= 1 - \left(\frac{A}{B}\right)^K = 1 - e^{-\mathsf{a}} \\ \frac{d^2 f(t)}{dt^2} &= 0. \end{aligned} \qquad \text{(A.166)}$$

A.8 ANALYSIS AND ESTIMATION

A.8.1 Fourier-transform effects

For practical reasons, we estimate the spectrum via the Fourier transform of the sequence of counts $\{Z_k\}$. A simple factor of T^{-2} connects the count-based and rate-based spectral estimates. This gives rise to an estimated spectrum whose expected value differs from that of the point-process spectrum, as we now proceed to show (Thurner et al., 1997).

As previously, consider a set of M counts, each of duration T, with $0 \leq k < M$. Define the Fourier transform of the counts via

$$X(n) \equiv \sum_{k=0}^{M-1} Z_k \, e^{-i2\pi kn/M}. \tag{A.167}$$

The estimate of the spectrum then becomes

$$\begin{aligned} \widehat{S}_Z(n) &= M^{-1} \, |X(n)|^2 \\ &= M^{-1} \sum_{k=0}^{M-1} \sum_{m=0}^{M-1} Z_k \, Z_m \, e^{i2\pi(k-m)n/M}, \end{aligned} \tag{A.168}$$

with an expected value

$$\mathrm{E}\Big[\widehat{S}_Z(n)\Big] = M^{-1} \sum_{k=0}^{M-1} \sum_{m=0}^{M-1} e^{i2\pi(k-m)n/M} \, \mathrm{E}[Z_k Z_m] \, . \tag{A.169}$$

We can express the correlation between the counts in terms of the spectrum of the point process itself by means of

$$\begin{aligned} \mathrm{E}[Z_k Z_m] &= \mathrm{E}\left[\int_{s=0}^{T} \int_{t=0}^{T} dN(s+kT)\, dN(t+mT)\right] \\ &= \int_{s=0}^{T} \int_{t=0}^{T} G\big[s-t+(k-m)T\big] \, ds\, dt \\ &= \int_{u=-T}^{T} \int_{v=|u|}^{2T-|u|} G\big[u+(k-m)T\big] \, \frac{du\, dv}{2} \\ &= \int_{u=-T}^{T} (T-|u|)\, G\big[u+(k-m)T\big] \, du \\ &= \int_{u=-T}^{T} (T-|u|) \int_{f=-\infty}^{\infty} S_N(f)\, e^{i2\pi f[u+(k-m)T]} \, df\, du. \end{aligned} \tag{A.170}$$

Finally, combining Eqs. (A.169) and (A.170) yields

$$\mathrm{E}\Big[\widehat{S}_Z(n)\Big] = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{m=0}^{M-1} e^{i2\pi(k-m)n/M} \int_{u=-T}^{T} (T-|u|)$$

$$
\begin{aligned}
&\times \int_{f=-\infty}^{\infty} S_N(f)\, e^{i2\pi f[u+(k-m)T]}\, df\, du \\
= \;& \frac{1}{M} \int_{f=-\infty}^{\infty} S_N(f) \left| \sum_{k=0}^{M-1} e^{ik2\pi(n/M+fT)} \right|^2 \\
&\times \int_{u=-T}^{T} (T-|u|)\, e^{i2\pi f u}\, du\, df \\
= \;& \frac{1}{\pi^2 M} \int_{f=-\infty}^{\infty} S_N(f)\, \frac{\sin^2(\pi n + M\pi f T)}{\sin^2(\pi n/M + \pi f T)}\, \frac{\sin^2(\pi f T)}{f^2}\, df \\
= \;& \frac{T}{\pi M} \int_{-\infty}^{\infty} S_N\!\left(\frac{x}{\pi T}\right) \frac{\sin^2(x)}{x^2} \frac{\sin^2(Mx)}{\sin^2(x+\pi n/M)}\, dx. \qquad \text{(A.171)}
\end{aligned}
$$

For a fractal-based point process in which $S_N(f)$ takes the form of Eq. (5.44a), we obtain

$$
\mathrm{E}\Big[\widehat{S}_Z(n)\Big] = \frac{\mathrm{E}[\mu]\,T}{\pi M} \int_{-\infty}^{\infty} \left[1 + (\pi f_S T)^{\alpha}\, |x|^{-\alpha}\right] \frac{\sin^2(x)}{x^2} \frac{\sin^2(Mx)}{\sin^2(x+\pi n/M)}\, dx. \qquad \text{(A.172)}
$$

To proceed further, we consider the case $0 < \alpha < 2$, which encompasses the vast majority of fractal-based point processes observed in practice, as discussed in Sec. 5.2.2. The fraction inside the integral in Eqs. (A.171) and (A.172) then only becomes important within the range $-\pi(n+1)/M < x < -\pi(n-1)/M$. For large values of M, this range becomes quite narrow. Substituting $y \equiv x + \pi n/M$ we obtain

$$
\begin{aligned}
\lim_{x\to-\pi n/M} \frac{\sin^2(Mx)}{\sin^2(x+\pi n/M)} &= \lim_{y\to 0} \frac{\sin^2[M(y-\pi n/M)]}{\sin^2(y)} \\
&= \lim_{y\to 0} \frac{\sin^2(My)}{\sin^2(y)} \\
&= M^2. \qquad \text{(A.173)}
\end{aligned}
$$

For large M we can therefore insert Eq. (A.173) into Eq. (A.172) to obtain

$$
\begin{aligned}
&\mathrm{E}\Big[\widehat{S}_Z(n)\Big] \\
&\approx \frac{\mathrm{E}[\mu]\,T}{\pi M} \int_{-\infty}^{\infty} \left[1 + (\pi f_S T)^{\alpha}\, |x|^{-\alpha}\right] \frac{\sin^2(x)}{x^2}\, M^2\, \delta(x+\pi n/M)\, dx \\
&= \frac{\mathrm{E}[\mu]\, MT}{\pi} \left[\frac{\sin^2(\pi n/M)}{(\pi n/M)^2}\right] \left[1 + (f_S MT/n)^{\alpha}\right], \qquad \text{(A.174)}
\end{aligned}
$$

which essentially reproduces the dominant term of Eq. (3.67).

Finally, for low frequencies such that $n \ll M$, the factor in large brackets in Eq. (A.174) approaches unity, so that we recover the canonical form of Eq. (5.44a). For other values of n, this factor presents a confounding effect in estimating the fractal exponent.